# Analytic $K$-theory

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## Contents

# 1 Banach algebras and $C^*$-algebras

Our purpose in this chapter is to review some facts from the theory of $C^*$-algebras that we are going to need for the purposes of $K$-theory. We mainly omit proofs.

All of our elgebras and vector spaces are over either the field, $\mathbb{R}$, of real numbers, or the field $\mathbb{C}$ of complex numbers. Except where otherwise noted, either ground field could be in play. We use the symbol $\mathbb{C}$ to denote the field we are working over- whether it is the real numbers or the complex numbers.

## 1.1 Banach Algebras

Recall that an algebra, $A$, over the field $\mathbb{C}$ is a vector space over the field $\mathbb{C}$ equipped with a multiplication $A \times A \to A$ which turns $A$ into a ring and is compatible with the scalar multiplication on $A$ as a vector space.

**Definition 1.1** We call an algebra $A$ a *normed algebra* if it is also a normed vector space, and the norm $\| - \|$ satisfies the inequality

$$\|xy\| \leq \|x\|\|y\|$$

for all $x, y \in A$. A normed algebra is called a *Banach algebra* if it is a Banach space (ie: it is complete).

A normed or Banach algebra $A$ is called *unital* if there is an element $1 \in A$ such that $\|1\| = 1$, and $x1 = 1x = x$ for all $x \in A$. In this case we write $\lambda = \lambda 1 \in A$ for each scalar $\lambda \in \mathbb{C}$.

Note that $\|0\| = 0$, so $1 \neq 0$, and the Banach algebra $\{0\}$ is not considered unital.

**Example 1.2** Let $X$ be a compact Hausdorff space. Then the algebra $C(X)$, consisting of all continuous functions $X \to \mathbb{C}$ is a unital Banach algebra. Addition and multiplication of functions are defined pointwise. The norm is defined by the formula

$$\|f\| = \sup\{|f(s)| \mid s \in X\}$$

Similarly, for any topological space, the algebra $C_b(X)$ consisting of all bounded continuous functions $X \to \mathbb{C}$ is a $C^*$-algebra.

**Example 1.3** Let $V$ be a normed vector space. Let $\mathcal{L}(V)$ be the set of bounded linear maps $V \to V$. Then $\mathcal{L}(V)$ is a unital Banach algebra.

Addition of maps is defined pointwise, and multiplication by composition. The norm is the *operator norm*

$$\|T\| = \sup\{\|T(x)\| \mid \|x\| \leq 1\}.$$

In particular, the algebra of $n \times n$ matrices, $M_n(\mathbb{C}) = \mathcal{L}(\mathbb{C}^n)$ is a Banach algebra.

The following result is an easy consequence of the definition.

**Proposition 1.4** *In a normed algebra $A$ the operations of addition and multiplication,*

$$A \times A \to A$$

*are continuous.* □

**Corollary 1.5** *Let $A$ be a normed algebra. Then the operations of addition and multiplication extend continuously to the completion, $\overline{A}$. The completion $\overline{A}$ is a Banach algebra.* □

This process of completion is useful for construction of examples. For most of our analysis to work, we really do need to work with Banach algebras rather than normed algebras.

Let us say an element $x$ in a unital Banach algebra $A$ is *invertible* if there is an element $x^{-1} \in A$ such that $x^{-1}x = xx^{-1} = 1$. We write $GL(A)$ for the group of invertible elements in a Banach algebra $A$.

**Proposition 1.6** *Let $A$ be a unital Banach algebra, and let $x \in A$. Suppose that $\|x\| < 1$. Then the element $1 - x$ is invertible.*

**Proof:** Define

$$S_n = 1 + x + x^2 + \cdots + x^n$$

Then the sequence $(S_n)$ is a Cauchy sequence in $A$, and therefore converges to some element $y \in A$. But

$$S_n(1-x) = (1-x) + (x - x^2) + \cdots + (x^n - x^{n+1}) = 1 - x^{n+1}$$

Hence

$$\|S_n(1-x) - 1\| = \| - x^{n+1}\| \leq \|x\|^{n+1}$$

so

$$\|y(1-x) - 1\| = \lim_{n\to\infty} \|S_n(1-x) - 1\| = 0$$

since $\|x\| < 1$. Therefore $y(1-x) = 1$. Similarly $(1-x)y = 1$. □

Note that completeness of the Banach algebra $A$ is vital to the above proposition.

**Corollary 1.7** *Let $A$ be a unital Banach algebra, let $x \in A$, and $\lambda \in \mathbb{C}$. Suppose that $\|x\| < \lambda$. Then the element $\lambda - x$ is invertible.* □

**Corollary 1.8** *The set $GL(A)$ is an open subset of the space $A$.* □

## 1.2 $C^*$-algebras

**Definition 1.9** Let $A$ be an algebra over the field $\mathbb{C}$. An *involution* on $A$ is a map $A \to A$, written $x \mapsto x^\star$, such that:

- $(\lambda x + \mu y)^\star = \overline{\lambda} x^\star + \overline{\mu} y^\star$ for all $x, y \in A$ and $\lambda, \mu \in \mathbb{K}$
- $(x^\star)^\star = x$ for all $x \in A$
- $(xy)^\star = y^\star x^\star$ for all $x, y \in A$

A number of properties about the involution are easily deduced from the above definition, such as the formula $0^\star = 0$, and $1^\star = 1$ if the algebra $A$ is unital.

**Definition 1.10** A normed algebra, $A$, with involution is called a *pre-$C^*$-algebra* if the *$C^*$-inequality*

$$\|x\|^2 \leq \|x^* x + y^* y\|$$

holds for all elements $x, y \in A$. It is called a $C^*$-algebra if it is also complete.

A norm on a Banach algebra with involution that satisfies the $C^*$-inequality is called a *$C^*$-norm.*

**Proposition 1.11** *Let $A$ be a pre-$C^*$-algebra. Then $\|x^\star\| = \|x\|$ for all $x \in A$.* □

Thus the involution on a pre-$C^*$-algebra, $A$, is continuous, and so extends to the completion, $\overline{A}$. It is easy to check that the completion of a pre-$C^*$-algebra is a $C^*$-algebra.

This process of completion is useful for construction of examples, but for our analysis to work, we really do need to work with $C^*$-algebras rather than pre-$C^*$-algebras.

**Proposition 1.12** *Let $A$ be a $C^*$-algebra. Then the $C^*$-*identity

$$\|x\|^2 = \|x^* x\|$$

*holds for all $x \in A$.*

*Let $A$ be a* complex *Banach algebra with involution such that the $C^*$-identity holds. Then $A$ is a $C^*$-algebra.* □

Most books on $C^*$-algebras only examine complex $C^*$-algebras, and so use the $C^*$-identity rather than the $C^*$-inequality in the definition.

**Example 1.13** Let $X$ be a compact topological space. Then the algebra $C(X)$ in example 1.2 is a $C^*$-algebra. The involution is defined by the formula

$$f^\star(s) = \overline{\varphi(s)}$$

**Example 1.14** Let $X$ be a locally compact Hausdorff space. Let $X^+$ be the one-point compactification, defined by adding a point, $\infty$, at infinity. Then, as above, we can form the $C^*$-algebra $C_0(X)$ of all continuous functions $\varphi\colon X^+ \to \mathbb{C}$ such that $\varphi(\infty) = 0$.

Observe that $C^*$-algebra $C_0(X)$ is unital if and only if the space $X$ is compact. In this case, $C_0(X) = C(X)$.

**Example 1.15** Let $H$ be a Hilbert space. Then the algebra of operators $\mathcal{L}(H)$ is a $C^*$-algebra. The involution is defined by taking the adjoint.

## 1.3 $*$-homomorphisms

**Definition 1.16** If $A$ and $B$ are $C^*$-algebras, a *$\star$-homomorphism* is an algebra homomorphism $\alpha\colon A \to B$ such that $\phi(x^\star) = \phi(x)^\star$ for all $x \in A$.

A $\star$-homomorphism, $\alpha$, between unital $C^*$-algebras is itself termed *unital* if $\alpha(1) = 1$.

**Theorem 1.17** *Any $*$-homomorphism is continuous, with closed image, and is bounded with norm at most one. Any injective $*$-homomorphism is an isometry.* □

It follows by the above that the norm on a $C^*$-algebra is determined by the rest of the algebraic structure. There are subtleties to this point which we will return to.

A $\star$-homomorphism is called an *isomorphism* of $C^*$-algebras if it is bijective. This terminology makes sense because of the following.

**Proposition 1.18** *The inverse of a bijective $*$-homomorphism is also an $*$-homomorphism.* □

Recall that for any Hilbert space $H$ we can form the $C^*$-algebra $\mathcal{L}(H)$ of all bounded linear operators from $H$ to itself.

**Definition 1.19** Let $A$ be a $C^*$-algebra. A *representation* of $A$ is a $\star$-homomorphism $\rho\colon A \to \mathcal{L}(H)$ for some Hilbert space $H$. A representation $\rho$ is called *faithful* if it is injective.

The following major result is called the (non-commutative) *Gelfand-Naimark theorem*.

**Theorem 1.20** *Every $C^*$-algebra has a faithful representation.* □

The following result is also fundamental in the theory of $C^*$-algebras; it is called the *commutative Gelfand-Naimark theorem.*

**Theorem 1.21** *Let $A$ be an Abelian $C^*$-algebra. Then there is a locally compact Hausdorff space $\hat{A}$ such that the $C^*$-algebra $A$ is isomorphic to the $C^*$-algebra $C_0(\hat{A})$.* □

Given a continuous map $f\colon X \to Y$, we have an induced map $f^*\colon C_0(Y) \to C_0(X)$ defined by the formula $f^*(\varphi) = \varphi \circ f$.

In fact, we have a converse.

**Proposition 1.22** *Let $X$ and $Y$ be locally compact Hausdorff spaces, and let $\phi\colon C_0(Y) \to C_0(X)$ be a $*$-homomorphism. Then there is a continuous map $f\colon X \to Y$ such that $\phi = f^*$.* □

Thus the spaces $X$ and $Y$ are homeomorphic if and only if the $C^*$-algebras $C_0(X)$ and $C_0(Y)$ are isomorphic. This fact, along with the commutative Gelfand-Naimark theorem, is the motivation behind considering the analysis of invariants of $C^*$-algebras to be a form of *non-commutative geometry.*

## 1.4 Spectra

Much of the theory of complex Banach algebras is based upon the following innocuous-seeming definition.

**Definition 1.23** Let $A$ be a unital complex Banach algebra, and let $x \in A$. Then we define the *spectrum*, *Spectrum*$(x)$, to be the set of complex numbers $\lambda \in \mathbb{C}$ such that the element $x - \lambda$ is not invertible.

**Example 1.24** Let $f \in C(X)$ be a continuous complex-valued function. Since multiplication is defined pointwise, the spectrum *Spectrum*$(f)$ is the image of the function $f$.

**Theorem 1.25** *The spectrum Spectrum$(x)$ is a non-empty compact subset of the set of complex numbers, $\mathbb{C}$, contained in the closed disc $\overline{D(0, \|x\|)}$.* □

**Definition 1.26** Let $A$ be a unital Banach algebra, and let $x \in A$. Then we define the *spectral radius*, $R_\sigma(x)$, to be:

$$\sup\{|\lambda| \mid \lambda \in \mathit{Spectrum}(x)\}$$

By the above theorem, the spectral radius $R_\sigma(x)$, is well-defined, and $R_\sigma(x) \leq \|x\|$. The following result is called the *spectral radius formula.*

**Theorem 1.27** *Let $A$ be a unital complex Banach algebra, and let $x \in A$. Then*

$$R_\sigma(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$$

□

**Definition 1.28** An element of a $C^*$-algebra $A$ is called *self-adjoint* if $x^\star = x$.

**Proposition 1.29** *Let $x$ be a self-adjoint element of a unital $C^*$-algebra. Then Spectrum$(x) \subseteq \mathbb{R}$.*

**Proof:** Let $\mu \in \mathbb{R}$. Then by the $C^*$-identity:

$$\|x \pm i\mu\|^2 = \|(x+i\mu)(x-i\mu)\| = \|x^2 + \mu^2\| \leq \|x\|^2 + |\mu|^2$$

Now, let $\lambda \in Spectrum(x)$. Then $\lambda \pm i\mu \in Spectrum(x \pm i\mu)$ so

$$|\lambda \pm i\mu|^2 \leq \|x\|^2 + |\mu|^2$$

Write $\lambda = \alpha + i\beta$ where $\alpha, \beta \in \mathbb{R}$. Then

$$|\alpha|^2 + |\beta \pm \mu|^2 \leq \|x\|^2 + |\mu|^2$$

so

$$|\alpha|^2 + 2|\beta||\mu| \leq \|x\|^2$$

For this inequality to hold for all $\mu \in \mathbb{R}$, we must have $\beta = 0$. □

**Example 1.30** Consider the disk

$$D = \{z \in \mathbb{C} \mid |z| \leq 1\}$$

We define the *disk algebra*, $H(D)$, to be the algebra of all continuous functions $D \to \mathbb{C}$ that are holomorphic on the interior of $D$. As a closed subalgebra of $C(D)$, the disk algebra $H(D)$ is a Banach algebra. It has an involution defined by the formula

$$f^\star(z) = \overline{f(\overline{z})}$$

Consider the function $f(z) = z$. Then $f \in H(D)$, and $f^\star = f$. However, we can easily check that $Spectrum(f) = D \not\subseteq \mathbb{R}$, so the disk algebra $H(D)$ is not a $C^*$-algebra.

**Definition 1.31** Let $A$ be a $C^*$-algebra. We call an element $x \in A$ *normal* if $x^\star x = xx^\star$.

**Example 1.32** The algebra of $2 \times 2$ complex matrices, $M_2(\mathbb{C})$, can be considered to be the algebra of bounded linear transformations $\mathbb{C} \oplus \mathbb{C} \to \mathbb{C} \oplus \mathbb{C}$. With the operator norm, the algebra $M_2(\mathbb{C})$ is thus a $C^*$-algebra according to example 1.15.

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $A^\star = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. We have the formulae

$$AA^\star = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad A^\star A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

So the element $A \in M_2(\mathbb{C})$ is not normal.

The following is easily deduced from the $C^*$-identity and the spectral radius formula.

**Proposition 1.33** *Let $A$ be a unital $C^*$-algebra. Let $x \in A$ be normal. Then we have spectral radius $R_\sigma(x) = \|x\|$.* □

**Corollary 1.34** *We have the formula*

$$\|x\| = R_\sigma(x^\star x)^{\frac{1}{2}}$$

*for any element $x \in A$.*

**Proof:** The element $x^\star x$ is self-adjoint and therefore normal. The result therefore follows from the $C^*$-identity and the above proposition. □

The above is another way of seeing that the norm on a unital complex $C^*$-algebra is determined completely by the algebraic structure.

# 2 Functional Calculus and Other Constructions

## 2.1 Functional Calculus

Let $A$ be a complex unital $C^*$-algebra. *Functional calculus* is a way to assign an element $f(x) \in A$ to a function $f$ and a normal element $x \in A$. This assignment should obey certain rules suggested by the notation.

The basic idea is summarised in the following theorem.

**Theorem 2.1 (Functional Calculus)** *Let $A$ be a complex unital $C^*$-algebra, and let $x \in A$ be a normal element. Then there is a unique $*$-homomorphism $C(Spectrum(x)) \to A$, written $f \mapsto f(x)$, with the following properties.*

- *Let $id\colon Spectrum(x) \to \mathbb{C}$ be the identity function. Then $id(x) = x$.*
- *Let $\lambda \in \mathbb{C}$. Let $c_\lambda\colon Spectrum(x) \to \mathbb{C}$ be the constant map onto $\lambda$. Then $c_\lambda(x) = \lambda$.*

□

The main idea in the proof is the fact that since the element $x \in A$ is normal, the unital $C^*$-algebra generated by the elements 1, $x$, and $x^*$-is commutative, and so isomorphic to a $C^*$-algebra of the form $C(X)$ by the Gelfand-Naimark theorem. It turns out that we can identify the space $X$ with the spectrum $Spectrum(x)$.

The fact that we have a $*$-homomorphism $C(Spectrum(x)) \to A$ tells us that the element $f(x)$ is just what we would expect when the element $f$ is a polynomial.

The following is also important.

**Theorem 2.2 (The Spectral Mapping Theorem)** *Let $x \in A$ be a normal element of a unital $C^*$-algebra. Then for any function $f \in C(Spectrum(x))$ we have the formula*

$$Spectrum(f(x)) = f[Spectrum(x)]$$

□

**Proposition 2.3** *Let $x \in A$ be a normal element of a complex unital $C^*$-algebra. Let $f \in Spectrum(f(x))$, and $g \in f[Spectrum(f(x))]$ be composable functions. Then*

$$(g \circ f)(x) = g(f(x))$$

**Proof:** The result is obvious for polynomials. The general result follows since the functional calculus is continuous (as a $*$-homomorphism), and the polynomial algebras are dense subsets of the relevant algebras of continuous functions by the Stone-Weierstrass theorem. □

The following result is proved similarly.

**Proposition 2.4** *Let $A$ be a complex unital $C^*$-algebra, and let $x, y \in A$ be normal elements such that $xy = yx$. Let $f \in C(Spectrum(x)$. Then $f(x)y = yf(x)$.* □

Finally, we also have a continuity result.

**Proposition 2.5** *Let $A$ be a complex unital $C^*$-algebra, let $S \subseteq \mathbb{C}$, and let $\Omega_S$ be the set of all normal elements of $A$ with spectrum contained in $S$.*

*Let $f\colon S \to \mathbb{C}$ be a continuous map. Then the map $f\colon \Omega_S \to A$ defined by writing $x \mapsto f(x)$ is continuous.* □

We now look at some results that can be proved by use of the functional calculus. We begin with an easy characterisation of self-adjoint elements.

**Proposition 2.6** *Let $x \in A$ be a normal element of a complex unital $C^*$-algebra. Then $x$ is self-adjoint if and only if $Spectrum(x) \subseteq \mathbb{R}$.*

**Proof:** Define $f\colon \mathbb{C} \to \mathbb{C}$ by $f(z) = \overline{z}$. Then $f(z) = \overline{\mathrm{id}(z)}$ for all $z \in \mathbb{C}$, so $f = \mathrm{id}^*$, and by the functional calculus, $f(x) = x^*$.

If $f$ is self-adjoint, it follows that $f(\lambda) = \lambda$ for all $\lambda \in Spectrum(x)$, and so $Spectrum(x) \subseteq \mathbb{R}$.

Conversely, if $Spectrum(x) \subseteq \mathbb{R}$, then $f(\lambda) = \lambda$ for all $\lambda \in Spectrum(x)$, and so $x^* = f(x) = \mathrm{id}(x) = x$. □

**Definition 2.7** Let $u \in A$ be a normal element of a unital $C^*$-algebra. Then say that $u$ is *unitary* when $u^\star u = uu^\star = 1$.

Note that any unitary element is normal.

**Proposition 2.8** *Let $u \in A$ be a normal element of a complex unital $C^*$-algebra. Let*

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$$

*be the unit circle in $\mathbb{C}$.*

*Then $x$ is unitary if and only if $Spectrum(u) \subseteq \mathbb{T}$.* □

The above proposition is proved in the same way as proposition 2.6; the precise details are left as an exercise.

Finally, note that there is a version of functional calculus that works for Banach algebras rather than $C^*$-algebras, or for non-normal elements of a $C^*$-algebra. However, instead of continuous functions defined on the spectrum of an element, we need to work with holomorphic functions.

**Theorem 2.9 (Holomorphic Functional Calculus)** *Let $A$ be a unital Banach algebra, and let $x \in A$. Let $U \subseteq \mathbb{C}$ be an open set that contains the spectrum $Spectrum(x)$, and let $f : U \to \mathbb{C}$ be a holomorphic function.*

*Let $\gamma$ be a closed contour that winds once anticlockwise around $Spectrum(x)$. Then we have a well-defined element*

$$f_h(x) = \frac{1}{2\pi i} \oint_\gamma f(z)(z - T)^{-1} f(z)\ dz \in A$$

*This construction has the following properties.*

- *Let $H(x)$ be the algebra of all holomorphic functions defined on a neighbourhood of the spectrum, $Spectrum(x)$. Then we have an algebra homomorphism $H(x) \to A$ defined by the formula $f \mapsto f_h(x)$.*
- *Let $p(t)$ be a complex polynomial. Then $p(x) = p_h(x)$.*
- *$Spectrum(f_h(x)) = f[Spectrum(x)]$.*
- *Let $A$ be a $C^*$-algebra, and let $x \in A$ be normal. Then $f_h(x) = f(x)$.*

□

## 2.2 Tensor Products

Let $V$ and $W$ be vector spaces over the field $\mathbb{C}$. Recall that we define the *tensor product* $V \odot W$ to be the vector space over $\mathbb{C}$ generated by elements written $v \otimes w$, where $v \in V$ and $w \in W$. The element $v \otimes w$ is called an *elementary tensor.*

The elementary tensors are required to satisfy the formulae

- $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$.
- $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$.
- $\lambda(v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w)$.

If the vector spaces $V$ and $W$ are finite-dimensional, with bases $\{e_1, \ldots, e_m\}$ and $\{e'_1, \ldots, e'_n\}$ respectively, then the tensor product $V \odot W$ is also finite-dimensional, with basis

$$\{e_i \otimes e'_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$$

Hence

$$\dim(V \odot W) = (\dim(V))(\dim(W))$$

Tensor products of vector spaces are characterised by a universal property.

**Proposition 2.10** *Let $V$ and $W$ be vector spaces. Then there is a bilinear map $B\colon V \times W \to V \odot W$, such that, given a vector space $U$ and a bilinear map $B'\colon V \times W \to U$, there is a unique linear map $\phi\colon V \odot W \to U$ such that $\phi B = B'$.* □

**Corollary 2.11** *The tensor product $V \odot W$ is the unique vector space, up to isomorphism, with the property given in the above proposition.* □

Now, let $A$ and $B$ be algebras with involution. Then the vector space $A \otimes B$ is also an algebra with involution. The multiplication and involution are defined by the formulae

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 b_1) \otimes (a_2 b_2)$$

and

$$(a \otimes b)^* = a^* \otimes b^*$$

respectively.

Suppose that $A$ and $B$ are $C^*$-algebras. Then there is no obvious way to define a $C^*$-norm on the tensor product $A \odot B$. Further, even if we do define such a norm, it is not in general unique, and the tensor product $A \odot B$ is not in general complete.

In the rest of this section, we look at one possible construction of a $C^*$-norm. There are others, and they are different in general, although this is quite hard to prove.

**Definition 2.12** We call a $C^*$-algebra $A$ *nuclear* if there is a unique $C^*$-norm on the tensor product $A \odot B$ for any $C^*$-algebra $B$.

The following is straightforward to check.

**Proposition 2.13** *Let $H_1$ and $H_2$ be Hilbert spaces. Then we can define an inner product on the tensor product $H_1 \odot H_2$ by the formula*

$$\langle \eta_1 \otimes \xi_1, \eta_2 \otimes \xi_2 \rangle = \langle \eta_1, \eta_2 \rangle \cdot \langle \xi_1, \xi_2 \rangle$$

□

The above inner product defines a norm on the tensor product $H_1 \odot H_2$ in the usual way. Except for the finite-dimensional case, the space $H_1 \odot H_2$ is not complete with respect to this norm.

We label the completion $H_1 \otimes H_2$. The completion $H_1 \otimes H_2$ is a Hilbert space, called the *Hilbert space tensor product* of $H_1$ and $H_2$. The *vector space tensor product* $H_1 \odot H_2$ is a dense subset.

Now, consider $C^*$-algebras $A$ and $B$. By the Gelfand-Naimark theorem, we can find faithful representations $\rho_1\colon A \to \mathcal{L}(H_1)$ and $\rho_2\colon B \to \mathcal{L}(H_2)$. We have an injective $*$-homomorphism $\rho_1 \otimes \rho_2\colon A \otimes B \to \mathcal{L}(H_1) \otimes \mathcal{L}(H_2)$ defined by the formula

$$(\rho_1 \otimes \rho_2)(a \otimes b) = \rho_1(a) \otimes \rho_2(b)$$

Now, let $T_1\colon H_1 \to H_1$ and $T_2\colon H_2 \to H_2$ be bounded linear operators. Then we have an injective $*$-homomorphism $\phi\colon \mathcal{L}(H_1) \otimes \mathcal{L}(H_2) \to \mathcal{L}(H_1 \hat{\otimes} H_2)$ defined by the formula

$$\phi(T_1 \otimes T_2)(\eta \otimes \xi) = T_1 \eta \otimes T_2 \xi$$

Hence $\phi(\rho_1 \otimes \rho_2)\colon A \otimes B \to \mathcal{L}(H_1 \hat{\otimes} H_2)$ is an injective $*$-homomorphism. Thus we can define a $C^*$-norm on $A \odot B$ by the formula

$$\|x\| = \|\phi(\rho_1 \otimes \rho_2)(x)\|$$

for all $x \in A \otimes B$.

**Definition 2.14** The $C^*$-algebra $A \otimes B$ defined by completing $A \odot B$ with respect to the above norm is called the *spatial tensor product* of $A$ and $B$.

**Proposition 2.15** *The above norm does not depend on the choice of faithful representations $\rho_1$ and $\rho_2$ or the Hilbert spaces $H_1$ and $H_2$.*

**Proof:** Let $H_1'$ and $H_2'$ be Hilbert spaces with faithful representations $\rho_1'\colon A \to \mathcal{L}(H_1)$ and $\rho_2'\colon B \to \mathcal{L}(H_2)$. Let $\phi'\colon \mathcal{L}(H_1') \otimes \mathcal{L}(H_2') \to \mathcal{L}(H_1' \otimes H_2')$ be the embedding defined as above.

Then, since the map $\phi'(\rho_1' \otimes \rho_2')$ is injective, we have a well-defined injective $*$-homomorphism

$$\alpha\colon\ im\,\phi'(\rho_1' \otimes \rho_2') \to im\,\phi(\rho_1 \otimes \rho_2)$$

defined by the formula

$$\alpha(\phi'(\rho_1' \otimes \rho_2')(x)) = \phi(\rho_1 \otimes \rho_2)(x)$$

As an injective $*$-homomorphism between $*$-subalgebras of $\mathcal{L}(H_1' \hat{\otimes} H_2')$ and $\mathcal{L}(H_1 \otimes H_2)$ respectively, we see that the map $\alpha$ is an isometry.

The result now follows. $\square$

We conclude with some examples.

**Example 2.16** Let $M_n(\mathbb{C})$ be the $C^*$-algebra of complex $n \times n$ matrices equipped with the operator norm (as operators on the Hilbert space $\mathbb{C}^n$).

Let $M_n(A)$ be the $*$-algebra of $n \times n$ matrices with entries in the $C^*$-algebra $A$. Let $e_{ij}$ be the $n \times n$ matrix with $(i,j)$-element equal to 1, and the other entries all zero. Then we can define an isomorphism $\phi\colon M_n(A) \to A \otimes M_n(\mathbb{C})$ by the formula

$$\phi(\sum_{i,j=1}^{n} a_{ij} e_{ij}) = \sum_{i,j=1}^{n} a_{ij} \otimes e_{ij}$$

By definition, the map $\phi$ is linear. It is straightforward to check that the map $\phi$ is compatible with the multiplication and involution; note that

$$e_{ij} e_{kl} = \begin{cases} e_{il} & j = k \\ 0 & j \neq k \end{cases}$$

and $e_{ij}^* = e_{ji}$.

Now, let $C \in M_n(A)$. Let $v \in A^n = A \oplus \cdots \oplus A$. Then we can consider $C$ as a bounded linear map $C\colon A^n \to A^n$ by matrix multiplication; we can equip $M_n(A)$ with the operator norm. If we do this, it is straightforward to check that $M_n(A)$ is a $C^*$-algebra.

Since the norm in a $C^*$-algebra is completely determined by the algebraic structure, we see that the tensor product $A \otimes M_n(\mathbb{C})$ is also a $C^*$-algebra. In particular, the $C^*$-algebra $M_n(\mathbb{C})$ is nuclear, and $A \hat{\otimes} M_n(\mathbb{C}) = A \otimes M_n(\mathbb{C})$.

Because of the above, we use the notations $M_n(A)$ and $A\hat{\otimes}M_n(\mathbb{C})$ interchangeably.

Now, let $X$ be a locally compact Hausdorff space, with one-point compactification $X^+$. Then we define $C_0(X \to A)$ to be the $C^*$-algebra of all continuous functions $\varphi\colon X \to A$ such that $\varphi(\infty) = 0$.

Addition and multiplication of functions are defined pointwise. The norm is defined by the formula

$$\|\varphi\| = \sup\{\|\varphi(s)\| \mid s \in X\}$$

and the involution is defined by the formula

$$\varphi^*(s) = \varphi(s)^*$$

**Proposition 2.17** *We have $C_0(X \to A) = C_0(X)\hat{\otimes}A$.* □

## 2.3 Complexification

In order to apply the theory of spectra and functional calculus to real $C^*$-algebras, we use a procedure for turning real $C^*$-algebras into complex $C^*$-algebras.

**Proposition 2.18** *The field of complex numbers, $\mathbb{C}$, is a real $C^*$-algebra.*

**Proof:** The field $\mathbb{C}$ is obviously a real algebra. There is a norm defined by taking the absolute value, and $\mathbb{C}$ is known to be complete under this norm. There is an involution defined by complex conjugation.

It remains to check the $C^*$-inequality. Let $w, z \in \mathbb{C}$. Then

$$|w|^2 = \overline{w}w \leq \overline{w}w + \overline{z}z = |\overline{w}w + \overline{z}z|$$

and the result follows. □

**Definition 2.19** Let $A$ be a real $C^*$-algebra. Then we define the *complexification* of $A$ to be the real $C^*$-algebra $A \otimes \mathbb{C}$.

**Proposition 2.20** *The complexification $A \otimes \mathbb{C}$ is a complex $C^*$-algebra.*

**Proof:** Note that $A \otimes \mathbb{C}$ is a complex vector space, with complex scalar multiplication defined by writing

$$\lambda(a \otimes \mu) = a \otimes (\lambda\mu) \qquad a \in A,\ \lambda, \mu \in \mathbb{C}.$$

It is easy to see that this complex scalar multiplication is compatible with the multiplication and norm, and that the involution is conjugate-linear. The result now follows once we check the $C^*$-identity. □

Observe that we have a canonical inclusion $i\colon A \hookrightarrow A \otimes \mathbb{C}$ defined by writing $a \mapsto a \otimes 1$. Let $a \in A$ and $\lambda \in \mathbb{R}$. Then

$$a \otimes \lambda = (\lambda a) \otimes 1 \in i[A].$$

The process of complexification is functorial.

**Proposition 2.21** *Let $A$ and $B$ be real $C^*$-algebras, and let $\alpha\colon A \to B$ be a $\star$-homomorphism. Then there is an induced $\star$-homomorphism between complex $C^*$-algebras, $\alpha \otimes 1\colon A \otimes \mathbb{C} \to B \otimes \mathbb{C}$, defined*

$$\alpha \otimes 1(x \otimes \lambda) = \alpha(x) \otimes \lambda$$

□

We now come to our applications to spectra and functional calculus.

**Definition 2.22** Let $A$ be a real unital $C^*$-algebra, and let $x \in A$. Then we define the *spectrum*, *Spectrum*$(x)$, to be the spectrum of the element $x \otimes 1 \in A \otimes \mathbb{C}$.

The following notion is due to Atiyah.

**Definition 2.23** For any subset, $X$, of the complex numbers we define the space of *Real*[1] continuous functions on $X$ by

$$C_{\mathbb{R}}(X) = \{f \in C(X) \mid f(\overline{x}) = \overline{f(x)} \text{ for all } x \in X\}$$

Here $C(X)$ denotes the set of *complex-valued* continuous functions on the space $X$.

**Proposition 2.24** *Let $x \in A$ be a normal element in a real unital $C^\star$-algebra $A$. Let $f \in C_{\mathbb{R}}(\mathit{Spectrum}(x))$ be a Real function defined on the spectrum of $x$. Then we have an element $f(x) \in A$ such that $f(x \otimes 1) = f(x) \otimes 1$.*

*Further, we have spectrum $\mathit{Spectrum}(f(x)) = f[\mathit{Spectrum}(x)]$.* □

The above construction has similar properties to functional calculus in the complex case. For instance, we have a $*$-homomorphism $C_{\mathbb{R}}(\sigma(a)) \to A$ defined by writing $f \mapsto f(a)$.

Holomorphic functional calculus can be similarly adapted.

## 2.4 Unitisations

Many naturally occurring and useful $C^*$-algebras are not unital. One way to deal with non-unital $C^*$-algebras is *unitisation*: a process that turns a $C^*$-algebra into a unital $C^*$-algebra.

**Definition 2.25** Let $A$ be a $C^*$-algebra. Then we define the *unitisation*, $A^+$, to be the direct sum $A \oplus \mathbb{C}$ as a vector space. Multiplication, norm, and involution are defined by the formulae

$$(x + \lambda)(y + \mu) = xy + \lambda y + \mu x + \lambda\mu$$

,

$$\|x + \lambda\|_{A^+} = \sup\{\|xy + \lambda y\|_A \mid y \in A, \|y\|_A \leq 1\}$$

and

$$(x + \lambda)^\star = x^\star + \overline{\lambda}$$

respectively.

[1] The capitalisation of the word 'Real' here is intentional.

**Proposition 2.26** *The $C^*$-algebra $A^+$ is a unital $C^*$-algebra. Further, $\|x\|_A = \|x+0\|_{A^+}$ for each element $x \in A$.*

**Proof:** It is a simple matter of axiom checking to prove that the unitisation $A^+$ is a unital $C^*$-algebra. Further, the inequality $\|x+0\|_{A^+} \leq \|x\|_A$ is clear from the definition of the norm $\| - \|_{A^+}$.

We thus need to prove the inequality $\|x\|_A \leq \|x+0\|_{A^+}$. This fact is obvious when $x = 0$. If $x \neq 0$, write $y = \frac{x^*}{\|x\|}$. Then $\|y\|_A = 1$, and $\|xy\|_A = \|x\|_A$ by the $C^*$-identity. The desired inequality $\|x\|_A \leq \|x+0\|_{A^+}$ now follows from the definition of the norm $\| - \|_{A^+}$. □

The unit in the $C^*$-algebra $A^+$ is of course defined by the formula $1 = 0 + 1$, where $0 \in x$ and $1 \in \mathbb{C}$. The above proposition means that we can regard the $C^*$-algebra $A$ as a subalgebra of the unitisation $A^+$. We then write $x = x + 0$ to denote the element of the unitisation $A^+$ corresponding to the element $x \in A$.

**Proposition 2.27** *Let $\alpha\colon A \to B$ be a $\star$-homomorphism. Then there is an induced unital $\star$-homomorphism $\alpha^+ \colon A^+ \to B^+$ defined by the formula*

$$\alpha^+(x+\lambda) = \alpha(x) + \lambda$$

□

**Example 2.28** Let $X$ be a locally compact Hausdorff space, with one point compactification $X^+ = X \cup \{\infty\}$. We can define the $C^*$-algebra $C_0(X)$ to be the algebra of continuous functions $f\colon X^+ \to \mathbb{C}$ such that $f(\infty) = 0$. The norm and involution are defined by the formulae

$$\|f\| = \sup\{|f(x)| \mid x \in X\}$$

and

$$f^\star(x) = \overline{f(x)}$$

respectively.

Then the one point compactification $C_0(X)^+$ is naturally isomorphic to the $C^*$-algebra $C(X^+)$.

**Definition 2.29** For two $C^*$-algebras $A$ and $B$ we define the *direct sum* $A \oplus B$ to be the direct sum of $A$ and $B$ as algebras, with norm and involution defined by the formulae

$$\|x \oplus y\| = \sup\{\|x\|, \|y\|\}$$

and

$$(x \oplus y)^\star = x^\star \oplus y^\star$$

respectively.

It is easy to verify that the direct sum $A \oplus B$ is a $C^*$-algebra. Note that the unitisation $A^+$ of a $C^*$-algebra is *not* the same as the direct sum $A \oplus \mathbb{C}$; the multiplication is defined differently.

**Proposition 2.30** *The $C^*$-algebras $A^+$ and $A \oplus \mathbb{C}$ are isomorphic if and only if the $C^*$-algebra $A$ is unital.*

**Proof:** If the $C^*$-algebra $A$ is non-unital, then the direct sum $A \oplus \mathbb{C}$ is also non-unital, and so cannot by isomorphic to the unital $C^*$-algebra $A^+$.

Conversely, suppose that the $C^*$-algebra $A$ is unital, with unit $e$. Then we can define an isomorphism of $C^*$-algebras $\phi\colon A \oplus \mathbb{C} \to A^+$ by the formula

$$\phi(x \oplus \lambda) = x - \lambda e + \lambda$$

□

Equipped with the idea of unitisations, we can define the spectrum of an element of a non-unital $C^*$-algebra.

**Definition 2.31** Let $A$ be a non-unital $C^*$-algebra, and let $x \in A$. Then we define the *spectrum*, $Spectrum(x)$, to be the spectrum of the element $x+0 \in A^+$.

Let $x \in A$, where the $C^*$-algebra $A$ is non-unital. Observe that the element $x + 0 \in A^+$ is not invertible. Hence the spectrum $Spectrum(x)$ always contains the zero element $0 \in \mathbb{C}$.

Finally, we look at functional calculus.

**Proposition 2.32** *Let $A$ be a $C^*$-algebra, and let $x \in A$ be normal. Let $f \in C(Spectrum(x))$ be such that $f(0) = 0$. Then $f(x) \in A$.*

**Proof:** The above definition gives us $f(x) \in A^+$. When $f(t)$ is a polynomial, the condition $f(0) = 0$ tells us that

$$f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t$$

so

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x \in A$$

Now, by the Stone-Weierstraß theorem, any $f \in C(Spectrum(x))$ such that $f(0) = 0$ is a limit of polynomials of the above form. The functional calculus is continuous, and the $C^*$-algebra $A$ is a closed subalgebra of the unitisation $A^+$. The result now follows. □

A similar result holds when $A$ is not normal, and we have a suitable holomorphic function defined in a neighbourhood of $Spectrum(x)$.

## 2.5 Some Lifting Results

Let $\alpha\colon A \to B$ be an epimorphism of $C^\star$-algebras. Let $Y$ be a topological space. In this section we consider results concerning the lifting of a map $f\colon Y \to B$ to a map $g\colon Y \to A$ such that $\alpha g = f$. These results are technical, but will be of vital importance to us later on.

**Lemma 2.33** *Let $\alpha\colon A \to B$ be an epimorphism of Banach spaces. Let $\{b_\lambda \mid \lambda \in \Lambda\}$ be a bounded subset of the Banach space $B$.*

*Then there is a bounded subset $\{a_\lambda \mid \lambda \in \Lambda\}$ of $A$ such that $\alpha(a_\lambda) = b_\lambda$ for all $\lambda$.*

**Proof:** By the open mapping theorem, we have a Banach space isomorphism

$$\theta\colon \frac{A}{\ker\,\alpha} \to B$$

defined by $\theta(\pi(a)) = \alpha(a)$ for all $a \in A$ [2]

So there is a bounded set $\{\pi(c_\lambda) \mid \lambda \in \Lambda\}$ in the quotient space $A/\ker\,\alpha$ for which $\alpha(c_\lambda) = b_\lambda$ for all $\lambda$. Boundedness means we can find a number $K$ such that $\|\pi(c_\lambda)\| \leq K$ for all $\lambda \in \Lambda$. Hence, for all $\lambda \in \Lambda$

$$\inf\{\|c_\lambda + k\| \mid k \in \ker\,\alpha\} \leq K$$

by definition of the quotient norm.

We can therefore find and elements $k_\lambda \in \ker\,\alpha$ such that

$$\|c_\lambda + k_\lambda\| \leq K + 1$$

for all $\lambda \in \Lambda$

Define $a_\lambda = c_\lambda + k_\lambda$. Then $\{a_\lambda \mid \lambda \in \Lambda\}$ is a bounded subset of $A$ such that $\alpha(a_\lambda) = b_\lambda$ for all $\lambda \in \Lambda$. □

Before stating our main result, we need a brief digression on partitions of unity.

**Definition 2.34** Let $X$ be a topological space. Let $\mathcal{U} = \{U_\lambda \mid \lambda \in \Lambda\}$ be an *open cover* of $X$, that is a collection of open subsets of $X$ such that $\cup_{\lambda\in\Lambda} U_\lambda = X$.

A *partition of unity subordinate to* $\mathcal{U}$ is a collection of continuous functions $\{\phi_\lambda\colon X \to [0,1] \mid \lambda \in \Lambda\}$ such that:

- $\phi_\lambda(x) = 0$ if $x \notin U_\lambda$.
- $\sum_{\lambda\in\Lambda} \phi_\lambda(x) = 1$ for all $x \in X$.

We need the following result from general topology.

**Theorem 2.35** *Let $X$ be a compact Hausdorff space. Then any open cover of $X$ has a partition of unity subordinate to it.* □

**Theorem 2.36** *Let $\alpha\colon A \to B$ be an epimorphism of $C^\star$-algebras. Let $Y$ be a compact metric space. Then the map*

$$\alpha_\star\colon C(Y) \otimes A \to C(Y) \otimes B$$

*defined by the formula $\alpha_*(f)(y) = \alpha(f(y))$ is also an epimorphism.*

**Proof:** Since $\alpha_\star$ is certainly a $*$-homomorphism, its image is closed in the $C^\star$-algebra $C(Y) \otimes B$. It therefore suffices to show that the image of the map $\alpha_\star$ is dense in the $C^\star$-algebra $C(Y) \otimes B$.

Let $f \in C(Y) \otimes B$ and let $\varepsilon > 0$. Then by continuity of the function $f$ we can find an open cover $\mathcal{U} = \{U_\lambda \mid \lambda \in \Lambda\}$ of $Y$ such that the inequality $\|f(x) - f(y)\| < \varepsilon$ is satisfied whenever $x$ and $y$ belong to the same open set $U_\lambda$.

[2] Here $\pi\colon A \to A/\ker\,\alpha$ denotes the obvious quotient map

Choose a point $x_\lambda \in U_\lambda$ for each element $\lambda \in \Lambda$. Then the function $f$ is bounded so the set $\{f(x_\lambda) \mid \lambda \in \Lambda\}$ is a bounded subset of the $C^\star$-algebra $B$. By lemma 2.33 we can find a bounded subset $\{a_\lambda \mid \lambda \in \Lambda\}$ of $A$ for which $\alpha(a_i) = f(x_i)$ for each $\lambda$.

By the above, we can find a partition of unity $\{\phi_\lambda \mid \lambda \in \Lambda\}$ subordinate to the open cover $\mathcal{U}$. Since the set $\{a_\lambda \mid \lambda \in \Lambda\}$ is bounded we can define a function $g \in C(Y) \otimes A$ by the formula

$$g(x) = \sum_{\lambda \in \Lambda} a_\lambda \phi_\lambda(x)$$

For any point $x \in Y$ we have

$$\begin{array}{rcl} \|(\alpha_\star g)(x) - f(x)\| & = & \|\sum_{\lambda\in\Lambda} f(x_\lambda)\phi_\lambda(x) - f(x)\| \\ & = & \|\sum_{\lambda\in\Lambda} \phi_\lambda(x)(f(x) - f(x_\lambda))\| \\ & \leq & \sum_{\lambda\in\Lambda} \phi_\lambda(x)\|f(x) - f(x_\lambda)\| \\ & < & \varepsilon \end{array}$$

since if $\phi_\lambda(x) \neq 0$ then $x \in U_\lambda$ so $\|f(x) - f(x_\lambda)\| < \varepsilon$.

Hence $\|\alpha_\star g - f\| \leq \varepsilon$.

The image of $\alpha_\star$ is therefore dense in the $C^\star$-algebra $C(Y) \otimes B$ and we are done. □

We can obviously apply the above to the space $[0,1]$ to obtain the following result concerning paths of elements.

**Corollary 2.37** *Let $\alpha\colon A \to B$ be a surjective $*$-homomorphism of $C^\star$-algebras. Let $\{b_t \mid t \in [0,1]\}$ be a path of elements of $B$.*

*Then there is a path $\{a_t \mid t \in [0,1]\}$ of elements of $A$ such that $\alpha a_t = b_t$ for all $t$.* □

# 3 Gradings and Clifford Algebras

## 3.1 The Category of Graded $C^\star$-algebras

**Definition 3.1** A *grading* on a given $C^\star$-algebra $A$ is a $C^\star$-algebra automorphism $\epsilon\colon A \to A$ such that $\alpha^2 = 1$.

If a $C^\star$-algebra $A$ is equipped with a grading, $\alpha$, we consider the grading to be part of the structure of the $C^\star$-algebra and refer to $A$ as a *graded $C^\star$-algebra* or a *$C^\star$-superalgebra*.

Observe that the identity map, $1\colon A \to A$, is a grading. It is called the *trivial grading*. Any (ungraded) $C^\star$-algebra may be considered to be a graded $C^\star$-algebra equipped with the trivial grading. Looked at in this way, the notion of a graded $C^\star$-algebra is a *generalisation* of the notion of a $C^\star$-algebra. It turns out to be the case that $K$-theory supports this point of view.

Another way of defining a grading on an object is to talk about *odd* and *even* elements.

**Definition 3.2** Let $A$ be a graded $C^\star$-algebra with grading $\alpha$. Then we define

$$A_{\text{even}} = \{a \in A \mid \alpha a = a\} \qquad A_{\text{odd}} = \{a \in A \mid \alpha a = -a\}$$

The following result is easy to see.

**Proposition 3.3** *The sets $A_{\text{even}}$ and $A_{\text{odd}}$ are closed subspaces of the $C^\star$-algebra $A$ such that $A = A_{\text{even}} \oplus A_{\text{odd}}$. Further, the product of two odd or two even elements is even, and the product of an odd and an even element is odd.* □

Perhaps more interesting, although just as straightforward, is the converse.

**Proposition 3.4** *Let $A$ be any $C^\star$-algebra. Suppose we have closed subspaces $A_{\text{even}}$ and $A_{\text{odd}}$ such that $A = A_{\text{even}} \oplus A_{\text{odd}}$, the product of two odd or two even elements is even, and the product of an odd and an even element is odd.*

*Define a map $\alpha\colon A \to A$ by writing*

$$\alpha(a+b) = a - b \textit{ for all alements } a \in A_{\text{even}},\ b \in A_{\text{odd}}$$

*Then the map $\alpha$ is the unique grading on the $C^\star$-algebra $A$ such that:*

$$A_{\text{even}} = \{a \in A \mid \alpha a = a\} \qquad A_{\text{odd}} = \{a \in A \mid \alpha a = -a\}$$

□

In some references the *degree* of an element of a $C^\star$-algebra is mentioned. It is merely another way of expressing whether the element is odd or even.

**Definition 3.5** Let $A$ be a graded $C^\star$-algebra. Then we define, for each element $a \in A$:

$$\deg(a) = \begin{cases} 0 & a \in A_{\text{even}} \\ 1 & a \in A_{\text{odd}} \end{cases}$$

The notion of degree can be useful for simplifying expressions in which the grading of a $C^\star$-algebra is involved.

**Definition 3.6** Let $A$ be a graded $C^\star$-algebra and let $a, b \in A$. Then we define the *graded commutator* of the elements $a$ and $b$ by the formula

$$[a,b] = ab - (-1)^{\deg(a)\ \deg(b)} ba$$

Note that the graded commutator $[a,b]$ is only defined when the elements $a, b \in A$ belong to the subset $A_{\text{even}} \cup A_{\text{odd}}$. We can define the graded commutator $[a,b]$ for other elements $a, b \in A$ by requiring it to be bilinear. This idea is frequently used when writing formulae in which the grading of a $C^\star$-algebra is involved.

**Definition 3.7** Let $A$ and $B$ be graded $C^\star$-algebras. A *morphism of graded $C^\star$-algebras* is a $C^\star$-algebra homomorphism $f\colon A \to B$ that takes odd elements to odd elements and even elements to even elements.

With this notion of morphism, the class of all graded $C^\star$-algebras is a category.

We define the subcategory of *unital* graded $C^\star$-algebras to be the category in which the objects are the graded unital $C^\star$-algebras, and the morphisms are graded $C^\star$-algebra morphisms $f\colon A \to B$ with $f(1) = 1$.

**Proposition 3.8** *Let $A$ be a graded $C^\star$-algebra. Let $x \in A$, and let $f \in C(Spectrum(x))$ be Real if $A$ is a real $C^*$-algebra, and extend to a holomorphic function in a neighbourhood of $Spectrum(x)$ if $x$ is not normal.*

*Suppose that $f(-\lambda) = -f(\lambda)$ for all $\lambda \in Spectrum(x)$. Then:*

- *$f(x)$ is odd if $x$ is odd*
- *$f(x)$ is even if $x$ is even*

□

## 3.2 Tensor Products

When considering graded $C^*$-algebras, we need to modify the tensor product we introduced earlier.

**Definition 3.9** Let $A$ and $B$ be graded $C^\star$-algebras. Then we define $A \hat{\otimes} B$ to be the Banach space with the same elements and norm as the spatial tensor product of $A$ and $B$.

We equip $A \hat{\otimes} B$ with involution, multiplication, and grading defined by the formulae:[3]

- $(a \otimes b)^\star = (-1)^{\deg(a)\ \deg(b)} a^\star \otimes b^\star$
- $(a \otimes b)(c \otimes d) = (-1)^{\deg(b)\ \deg(c)} (ac \otimes bd)$
- $\gamma(a \otimes b) = \alpha(a) \otimes \beta(b)$

**Proposition 3.10** *The object $A \hat{\otimes} B$ is a graded $C^\star$-algebra,* □

The graded $C^\star$-algebra $A \hat{\otimes} B$ is called the (spatial) *graded tensor product* of the graded $C^\star$-algebras $A$ and $B$. When it is implicit that we are dealing with graded $C^\star$-algebras (as in nearly all of this thesis), we will write $A \otimes B$ rather than $A \hat{\otimes} B$ and refer to our construction as just the *tensor product* of the $C^\star$-algebras $A$ and $B$.[4]

Let us look at some examples of graded tensor products.

**Definition 3.11** We write $M_n(\mathbb{F})$ for the $C^\star$-algebra of $n$ by $n$ matrices over the field $\mathbb{F}$, equipped with the trivial grading.

**Proposition 3.12** *Let $A$ be a graded $C^\star$-algebra over the field $\mathbb{F}$. Then*

$$M_n(\mathbb{F}) \otimes A = M_n(A)$$

*where $M_n(A)$ is the set of $n \times n$ matrices with entries in the graded $C^\star$-algebra $A$. Here we say that a matrix in the $C^\star$-algebra $M_n(A)$ is* odd *when every element is odd, and* even *when every element is even.* □

[3] Each formula appearing in this definition is only defined on a subset of the tensor product $A \hat{\otimes} B$ but can easily be extended by linearity to cover the whole space.

[4] This terminology makes sense because $A \hat{\otimes} B = A \otimes B$ when the $C^\star$-algebras $A$ and $B$ are both trivially graded.

**Proposition 3.13** *Define $C_0(X \to A)$ to be the $C^\star$-algebra of continuous functions $X \to A$ converging to zero at infinity. Define a grading on the $C^\star$-algebra $C_0(X \to A)$ by saying that a function is* odd *when every element in its image is odd, and* even *when every element in its image is even.*

*Then*

$$C_0(X)\hat{\otimes}A = C_0(X \to A)$$

□

## 3.3 Clifford Algebras

In this section we look at a particular class of examples of finite-dimensional graded $C^\star$-algebras.

**Definition 3.14** Let $p,\, q \in \mathbb{N}$.[5] Then we define the $(p,q)$-*Clifford algebra*, $\mathbb{F}_{p,q}$, to be the algebra over the field $\mathbb{F}$[6] generated by elements

$$\{\ e_1, \ldots, e_p,\ f_1, \ldots, f_q\ \}$$

which pairwise anti-commute and satisfy the formulae:

$$e_i^2 = 1 \qquad f_j^2 = -1$$

We can turn the algebra $\mathbb{F}_{p,q}$ into a Hilbert space by decreeing that the canonical basis defined by taking products of the generators is orthonormal.

Any element of the algebra $\mathbb{F}_{p,q}$ then acts as a linear operator on the space $\mathbb{F}_{p,q}$ by multiplication (on the left). Clearly no two elements of the algebra $\mathbb{F}_{p,q}$ act on the space $\mathbb{F}_{p,q}$ as the same operator. It is easy to see that the space $\mathbb{F}_{p,q}$ is finite-dimensional, with dimension $2^{p+q}$, so these operators are bounded.

We therefore have a faithful representation

$$\rho\colon \mathbb{F}_{p,q} \to \mathcal{L}(\mathbb{F}_{p,q})$$

Using this representation we can consider the algebra $\mathbb{F}_{p,q}$ to be a subalgebra of the algebra $\mathcal{L}(\mathbb{F}_{p,q})$ of bounded linear maps from the Hilbert space $\mathbb{F}_{p,q}$ to itself.

**Proposition 3.15** *The algebra $\mathbb{F}_{p,q}$ is a $C^\star$-algebra. We can define a grading by saying that each of the generators is odd.* □

We end this section by looking at a few elementary properties of Clifford algebras. We merely sketch the proofs.

**Proposition 3.16**

$$\mathbb{F}_{p,q} \otimes \mathbb{F}_{r,s} \cong \mathbb{F}_{p+r,q+s}$$

**Proof:** This is easy to check by looking at the generators. □

[5] Here we include zero amongst the natural numbers

[6] Recall that the field $\mathbb{F}$ is either the field of real numbers, $\mathbb{R}$, or the field of complex numbers, $\mathbb{C}$.

**Proposition 3.17** *The Clifford algebra $\mathbb{F}_{1,1}$ is isomorphic to the algebra $M_2(\mathbb{F})$ equipped with the grading*

$$M_2(\mathbb{F})_{\text{even}} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{F} \right\}$$
$$M_2(\mathbb{F})_{\text{odd}} = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \mid b, c \in \mathbb{F} \right\}$$

**Proof:** The algebra $M_2(\mathbb{F})$ has generators

$$e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

such that

$$e^2 = 1, \; f^2 = -1, \; ef = -fe$$

The result now follows. □

When we eventually talk about the Bott periodicity theorem, the following results will also be useful.

**Proposition 3.18**

$$\mathbb{C}_{1,0} \cong \mathbb{C}_{0,1}$$

**Proof:** Let $e$ be the generator of the algebra $\mathbb{C}_{1,0}$. Define $f = ie$. Then the algebra $\mathbb{C}_{1,0}$ is generated by the element $f$. However, $f^2 = -1$ and so the result follows. □

**Corollary 3.19**

$$\mathbb{C}_{2,0} \cong \mathbb{C}_{1,1} \cong \mathbb{C}_{0,2}$$

**Proof:**

$$\begin{array}{rcll} \mathbb{C}_{2,0} & \cong & \mathbb{C}_{1,0} \otimes \mathbb{C}_{1,0} & \text{by proposition 3.16} \\ & \cong & \mathbb{C}_{0,1} \otimes \mathbb{C}_{1,0} & \text{by the above} \\ & \cong & \mathbb{C}_{1,1} & \text{by proposition 3.16} \end{array}$$

The proof that $\mathbb{C}_{0,2} = \mathbb{C}_{1,1}$ is similar. □

Similarly, in the real case we have:

**Proposition 3.20**

$$\mathbb{R}_{4,0} \simeq \mathbb{R}_{0,4}$$

**Proof:** Let $\mathbb{R}_{4,0}$ have generators $e_1$, $e_2$, $e_3$, $e_4$. Write $f_1 = e_2e_3e_4$, $f_2 = e_1e_3e_4$, $f_3 = e_1e_2e_4$, $f_4 = e_1e_2e_3$.

Then $\mathbb{R}_{4,0}$ is generated by the elements $f_1$, $f_2$, $f_3$, and $f_4$. But $f_i^2 = -1$ for all $i$ and $f_if_j = -f_jf_i$ whenever $i \neq j$ so the result follows. □

**Corollary 3.21**

$$\mathbb{R}_{8,0} = \mathbb{R}_{4,4} \simeq \mathbb{R}_{0,8}$$

# 4 Supersymmetries and $K$-theory

## 4.1 Supersymmetries and Odd Involutions

**Definition 4.1** Let $A$ be a graded unital $C^\star$-algebra. We define a *supersymmetry*, $x$, in the algebra $A$ to be an odd self-adjoint involution, that is to say an element $x \in A_{\text{odd}}$ such that $x = x^\star$ and $x^2 = 1$.

We write the set of supersymmetries in the $C^\star$-algebra $A$ as $SS(A)$.

The set $SS(A)$ is a subspace of the normed space $A$. For points $x, y \in SS(A)$ let us write $x \simeq y$ to mean that $x$ and $y$ are in the same path-component of the space $SS(A)$.

**Definition 4.2** We define $OI(A)$ to be the set of odd involutions of the algebra $A$, that is the set of all elements $a \in A_{\text{odd}}$ such that $a^2 = 1$

Observe that the space of supersymmetries $SS(A)$ is a subspace of the space of involutions $OI(A)$.

**Lemma 4.3** *The space $SS(A)$ is a strong deformation retraction of the space $OI(A)$.*

**Proof:** We want to define a continuous map $f\colon OI(A) \to SS(A)$ and a homotopy $H\colon OI(A) \times [0,1] \to OI(A)$ such that $H(-,0) = f$, $H(-,1) = 1_{OI(A)}$, and $H(x,t) = x$ whenever $x \in SS(A)$ and $t \in [0,1]$.

Let $x \in OI(A)$. Define, using functional calculus

$$f(x) = (x^\star x)^{\frac{1}{4}} x (x^\star x)^{-\frac{1}{4}}$$

Then $f(x)$ is odd, and $f(x)^2 = 1$. But we may also write

$$\begin{aligned} f(x) &= (xx^\star)^{-\frac{1}{4}} x (x^\star x)^{-\frac{1}{4}} && \text{since } (x^\star x)^{-1} = xx^\star \\ &= x(x^\star x)^{-\frac{1}{2}} && \text{by proposition } \textbf{??} \end{aligned}$$

Hence $f(x)^\star = (x^\star x)^{-\frac{1}{2}} x^\star$ and so:

$$f(x) f(x)^\star = f(x)^\star f(x) = 1$$

If we recall that the element $f(x)$ is odd, and $f(x)^2 = 1$ we have proved that $f(x) \in SS(A)$. So we have defined a continuous map $f\colon OI(A) \to SS(A)$.

Define a map $H\colon OI(A) \times [0,1] \to OI(A)$ by:

$$H(x,t) = (t + (1-t)x^\star x)^{\frac{1}{4}} x (t + (1-t)x^\star x)^{-\frac{1}{4}}$$

Then $H$ is a homotopy between the functions $f$ and $1_{OI(A)}$. Further, if $x \in SS(A)$ then $x^\star x = 1$ and so $H(x,t) = x$ for all points $t \in [0,1]$. $\square$

**Lemma 4.4** *Let $A$ be any graded unital $C^\star$-algebra. Let $x, y \in SS(A)$ be supersymmetries, and suppose that $\|x - y\| < 2$. Then $x \sim y$.*

**Proof:** Let $z = \frac{1}{2}(1 + xy)$. Observe that:

$$\|z - 1\| \leq \frac{1}{2}\|x - y\|\|y\| < 1$$

Thus we have a path of even invertible elements:

$$z_t = (1 - t) + tz \qquad t \in [0, 1]$$

Let $w_t = z_t^{-1} x z_t$. Then $(w_t)$ is a path of odd involutions from the point $w_0 = x$ to the point $w_1 = z^{-1}xz$. But $z = \frac{1}{2}(1 + xy)$ so

$$xz = \frac{1}{2}(x + y) = zy$$

Therefore $y = z^{-1}xz$ and we have a path of odd involutions from $x$ to $y$. By lemma 4.3 this gives us a path of supersymmetries from $x$ to $y$. □

## 4.2 Semigroups and Grothendieck Completion

Recall that a set $V$ is a *semigroup* if it is equipped with an associative operation $*: V \times V \to V$. Given semigroups $V$ and $W$, a map $\gamma: V \to W$ such that $\gamma(v \star w) = \gamma(v) \star \gamma(w)$ is called a *semigroup homomorphism*.

We call a semigroup $V$ *abelian* if $x \star y = y \star x$ for all $x, y \in V$. Actually, when $V$ is abelian, we usually use the notation $+$ rather than $\star$ to denote the operation.

Thus an abelian semigroup is similar to an abelian group, but lacks an identity element and inverses. The following is easy to check.

**Proposition 4.5** *Let $V$ be an abelian semigroup. We can define an equivalence relation, $\sim$, on the product $V \times V$ by writing $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 + y_2 + z = x_2 + y_1 + z$.* □

We write $[x, y]$ to denote the equivalence class of the pair $(x_1, y_1)$. We think of the equivalence class $[x, y]$ as a way of making sense of the "formal difference" $x - y$. Note that with this notation

$$[x, y] = [x + z, y + z]$$

for all $x, y, z \in V$.

**Proposition 4.6** *The set of equivalence classes $G(V) = \{[x, y] \mid x, y \in V\}$ is an abelian group, with the operation*

$$[x_1, y_1] + [x_2, y_2] = [x_1 + y_1, x_2 + y_2].$$

**Proof:** It is straightforward to check that the operation is well-defined and associative.

Pick $z \in V$. Define $0 = [z, z]$. Observe, for all $[x, y] \in G(V)$, we have

$$[x, y] + 0 = [x + z, y + z] = [x, y]$$

so 0 is an identity element for $G(V)$. Note that, by uniqueness of an identity in an abelian group, it follows that $[z', z'] = 0$ for all $z' \in V$.

Observe

$$[x, y] + [y, x] = [x + y, y + x] = 0$$

as $V$ is abelian. Hence we have inverse $-[x, y] = [y, x]$ and we are done. □

Let $y \in V$. Then it is easy to see that the map $\gamma_V : V \to G(V)$ defined by the formula $\gamma_V(x) = [x + y, y]$ is a semigroup homomorphism independent of the choice of $V$.

**Definition 4.7** We call the group $G(V)$ the *Grothendieck group* of $V$. The homomorphism $\gamma_V : V \to G(V)$ is called the *Grothendieck map.*

The Grothendieck group can be characterised by a universal property.

**Proposition 4.8** *Let $V$ be an abelian group, and let $H$ be an abelian group. Let $\phi\colon V \to H$ be a semigroup homomorphism. Then there is a unique homomorphism $\psi\colon G(V) \to H$ such that $\psi \circ \gamma_V = \phi$.*

*Further, the Grothendieck group is the unique group, up to isomorphism, with this property.* □

This universal property is more convenient than the definition for recognising Grothendieck groups, and is used in the following two examples.

**Example 4.9** The Grothendieck completion of the semigroup of natural numbers is isomorphic to the abelian group $\mathbb{Z}$ of integers.

**Example 4.10** The set $\mathbb{Z}\backslash\{0\}$, with the operation of multiplication, is an abelian semigroup. The Grothendieck completion is isomorphic to the group $\mathbb{Q}\backslash\{0\}$ of non-zero rational numbers.

**Example 4.11** Consider the semigroup $\mathbb{N} \cup \{\infty\}$, where $n + \infty = \infty$ for all $n \in \mathbb{N}$. Then for any $m, n \in \mathbb{N}$, we have

$$[m, n] = [m + \infty, n + \infty] = [\infty, \infty] = 0.$$

So the Grothendieck completion of $\mathbb{N} \cup \{\infty\}$ is $\{0\}$.

## 4.3 The Group $K_1(A)$

For a topological space $X$, let us write $\pi_0(X)$ to denote the set of path-components. Given a graded unital $C^*$-algebra $A$, let us write $SS_n(A) = SS(M_n(A))$. Then we define $V_1(A)$ to be the union

$$V_1(A) = \bigcup_{n \in \mathbb{N}} \pi_0 SS_n(A).$$

Let us write $\langle x \rangle$ to denote the path-component in $V_1(A)$ of a supersymmetry $x \in SS_n(A)$. The following is easy to check.

**Lemma 4.12** *Suppose $SS(A) neq \emptyset$. The set $V_1(A)$ is a semigroup, with operation defined by the formula*

$$\langle x\rangle + \langle y\rangle = \langle x \oplus y\rangle$$

□

Let $G_1(A)$ be the Grothendieck completion of the semigroup $V_1(A)$. Then elements of the group $G_1(A)$ are equivalence classes $[\langle x\rangle, \langle y\rangle]$ where $x$ and $y$ are supersymmetries in matrices over $A$. To simplify notation, write

$$\langle x\rangle - \langle y\rangle = [\langle x\rangle, \langle y\rangle].$$

**Definition 4.13** Let $A$ be a unital $C^*$-algebra, and suppose $SS(A) \neq \emptyset$. Then we define the $K$-theory group

$$K_1(\mathcal{A}) = \{\langle x\rangle - \langle y\rangle \mid x, y \in SS_n(A), n \in \mathbb{N}\}$$

We will see later on how to remove the restriction to the case where $A$ is unital with a reference supersymmetry.

**Example 4.14** Clifford algebra.

## 4.4 Reference Supersymmetries

In this section we give an alternative definition of the group $K_1(A)$ where the Grothendieck construction is avoided.

Let $A$ be a unital graded $C^*$-algebra. Suppose we have a given *reference supersymmetry* $E \in SS(A)$. Write

$$E_n = E \oplus E \oplus \cdots \oplus E = \begin{pmatrix} E & & 0 \\ & \ddots & \\ 0 & & E \end{pmatrix} \in SS_n(A).$$

**Proposition 4.15** *Let $x, y \in SS_n(A)$. Suppose that there is an even unitary $u \in A$ such that $ux = yu$ and $uE = Eu$. Then $x \oplus E_n \sim E_n \oplus y$.*

**Proof:** For each point $\theta \in [0, \frac{\pi}{2}]$ let us define:

$$r_\theta = \begin{pmatrix} \cos\theta & u^\star \sin\theta \\ -u\sin\theta & \cos\theta \end{pmatrix}$$

Observe that:

$$r_\theta^{-1} = \begin{pmatrix} \cos\theta & -u^\star \sin\theta \\ u\sin\theta & \cos\theta \end{pmatrix} = r_\theta^\star$$

so that $(r_\theta)$ is a path of even unitary elements in the morphism set $Hom(A, B)$.

The equation

$$z_\theta = r_\theta \begin{pmatrix} x & 0 \\ 0 & E \end{pmatrix} r_\theta^{-1}$$

therefore defines a path of supersymmetries between the points $x \oplus E_n$ and $E_n \oplus y$. □

**Proposition 4.16** *Let $x, x' \in SS_m(A)$, and $y, y' \in SS_n(Q)$ be supersymmetries. Suppose that $x \oplus y \sim x' \oplus y'$. Then $y \oplus x \sim y' \oplus x'$.*

**Proof:** Let

$$\begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix} \qquad t \in [0,1]$$

be a path in the space $SS_{m+n}(A)$ from the point $x \oplus y$ to the point $x' \oplus y'$.

Then we can define a path

$$\begin{pmatrix} d_t & c_t \\ b_t & a_t \end{pmatrix} \qquad t \in [0,1]$$

in the space $SS_{m+n}(A)$ from the point $y \oplus x$ to the point $y' \oplus x'$. □

**Lemma 4.17** *Let $x \in SS_n(A)$. Then $x \oplus -E_n x E_n \sim E_{2n}$.*

**Proof:** Define

$$R_\theta = \begin{pmatrix} x\cos\theta & E_n \sin\theta \\ E_n \sin\theta & -E_n x E_n \cos\theta \end{pmatrix} \text{ for } \theta \in [0, \frac{\pi}{2}]$$

Then $R_\theta$ is a path of supersymmetries between

$$x \oplus -E_n x E_n = \begin{pmatrix} x & 0 \\ 0 & -E_n x E_n \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & E_n \\ E_n & 0 \end{pmatrix}$$

Taking $x = E_n$ and recalling that $E_n \sim -E_n$ we have a path of supersymmetries between

$$E_{2n} = \begin{pmatrix} E_n & 0 \\ 0 & E_n \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & E_n \\ E_n & 0 \end{pmatrix}$$

Therefore $x \oplus -E_n x E_n \sim E_{wn}$. □

**Definition 4.18** Given supersymmetries $x \in SS_m(A)$ and $y \in SS_n(B)$ we write $x \sim_E y$ whenever we have the relation $x \oplus E_{k+n} \sim y \oplus E_{k+m}$ for some $k$. We define the set $K_1^{(E)}(A)$ to be the collection of equivalence classes, $\langle x \rangle_E$, of supersymmetries, $x$, in the spaces $SS_n(A)$.

**Theorem 4.19** *The set $K_1^{(E)}(A)$ equipped with the operation*

$$\langle x \rangle_E + \langle y \rangle_E = \langle x \oplus y \rangle_E \text{ for all supersymmetries } x \text{ and } y$$

*is an abelian group.*

*We have identity element $\langle E \rangle_E$, and inverses defined by the formula $-\langle x \rangle_E = \langle -ExE \rangle_E$ for all supersymmetries $x$.*

**Proof:** Suppose that $\langle x \rangle_E = \langle x' \rangle_E$ and $\langle y \rangle_E = \langle y' \rangle_E$ for supersymmetries $x$, $x'$, $y$, and $y'$ in the $C^\star$-category $\mathcal{A}$.

Then we have the relation $x \oplus E_{n+k} \sim x' \oplus E_{m+k}$ for some $k$.

Hence

$$x \oplus E \oplus y \oplus E \sim x \oplus y \oplus E \sim E \oplus x' \oplus y' \oplus E$$

by proposition 4.16, so $\langle x \oplus y \rangle_E = \langle x' \oplus y' \rangle_E$. This establishes that the operation is well-defined.

Also, for any supersymmetries $x$, $y$, and $z$, it is clear that $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ which makes the operation associative.

Now, let $x \in SS_m(A)$ and $y \in SS_n(B)$ respectively. Define

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_{m+n}(A)$$

then clearly $S$ is an even unitary morphism for which $SE_{2n} = E_{2n}S$ and $S(x \oplus y) = (y \oplus x)S$.

Thus by lemma 4.15 we know that $x \oplus y \oplus E_{2n} \sim E_{2n} \oplus y \oplus x$. Hence $\langle x \oplus y \rangle_E = \langle y \oplus x \rangle_E$ and our operation is commutative.

Finally, by definition, $\langle x \oplus E \rangle_E = \langle x \rangle_E$ for any supersymmetry $x$, meaning we have identity element $\langle E \rangle_E$. Existence of inverses follows immediately from lemma 4.17. □

**Theorem 4.20** *Let $A$ be a unital $C^*$-algebra equipped with a reference supersymmetry, $E$. Then we have an isomorphism*

$$K_1(A) \cong K_1^{(E)}(A)$$

**Proof:** Try to define a map $f\colon K_1(A) \to K_1^{(E)}(A)$ by writing

$$f(\langle x \rangle - \langle y \rangle) = \langle x \rangle_E - \langle y \rangle_E$$

for all supersymmetries $x, y \in SS_n(A)$.

Suppose that $\langle x \rangle - \langle y \rangle = \langle \overline{x} \rangle - \langle \overline{y} \rangle$. Then:

$$\begin{array}{rrcl} & \langle x \rangle + \langle y' \rangle & = & \langle x' \rangle + \langle y \rangle \\ \Rightarrow & \langle x \oplus y' \rangle & = & \langle x' \oplus y \rangle \\ \Rightarrow & \langle x \oplus y' \rangle_E & = & \langle x' \oplus y \rangle_E \\ \Rightarrow & \langle x \rangle_E - \langle y \rangle_E & = & \langle x' \rangle_E - \langle y' \rangle_E \end{array}$$

So our map $f$ is well-defined. By construction it is a homomorphism. Let $\langle x \rangle_E$ be an element of the group $K_1^{(E)}(A)$. Then $f(\langle x \rangle - \langle E \rangle) = \langle x \rangle_E$ so the map $f$ is also surjective.

Now suppose that $f(\langle x \rangle - \langle y \rangle) = 0$. Then:

$$\begin{array}{rrcl} & \langle x \rangle_E & = & \langle y \rangle_E \\ \Rightarrow & x \oplus E & \sim & E \oplus y \oplus E \\ \Rightarrow & \langle x \oplus E \rangle & = & \langle E \oplus y \oplus E \rangle \\ \Rightarrow & \langle x \rangle + \langle E \rangle & = & \langle E \rangle + \langle y \rangle + \langle E \rangle \\ \Rightarrow & \langle x \rangle - \langle y \rangle & = & 0 \end{array}$$

So the map $f$ is also injective and we are done. □

**Corollary 4.21** *Up to isomorphism the group $K_1^{(E)}(\mathcal{A})$ does not depend on choice of reference supersymmetry.* □

# 5 Elements of Homotopy Theory

## 5.1 Categories and Functors

**Definition 5.1** A *category*, $\mathcal{C}$, consists of a collection (not necessarily a set), $Ob(\mathcal{C})$ of —em objects, and for any two objects $a, b \in Ob(\mathcal{C})$, a *morphism set* $Hom(a,b)_{\mathcal{C}}$, along with a *composition*

$$Hom(b,c)_{\mathcal{C}} \times Hom(a,b)_{\mathcal{C}} \to Hom(a,c)_{\mathcal{C}} \qquad (\beta, \alpha) \mapsto \beta \circ \alpha$$

such that:

- Composition of morphisms is *associative*, ie: $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$ for all $\alpha \in Hom(a,b)_{\mathcal{C}}$, $\beta \in Hom(b,c)_{\mathcal{C}}$ and $\gamma \in Hom(b,c)_{\mathcal{C}}$.
- For each object $a \in Ob(\mathcal{C})$, there is an *identity morphism* $\mathrm{id}_a \in Hom(a,a)_{\mathcal{C}}$ such that for all $\alpha \in Hom(a,b)_{\mathcal{C}}$, we have $\alpha \circ \mathrm{id}_a = \alpha = \mathrm{id}_b \circ \alpha$.

When the category $\mathcal{C}$ is apparent, we sometimes write just $Hom(a,b)$ rather than $Hom(a,b)_{\mathcal{C}}$ for the morphism sets.

There are numerous examples of categories in mathematics. Here are some that are useful to us.

- The category Set has, as objects, the collection of all sets. Given sets $A$ and $B$, the morphism set $Hom(A,B)$ consists of all functions from $A$ to $B$.
- The category Grp of all groups and group homomorphisms.
- The category Ab of all abelian groups and group homomorphisms.
- The category SAb of all abelian semigroups and semigroup homomorphisms.
- The category $C^*$ of all $C^*$-algebras and $*$-homomorphisms.
- The category $\hat{C^*}$ of all graded $C^*$-algebras and graded $*$-homomorphisms.
- The category Top of all topological spaces and continuous maps.

**Definition 5.2** Let $\mathcal{C}$ be a category. An *isomorphism* in $\mathcal{C}$ is a morphism $\alpha \in Hom(a,b)_{\mathcal{C}}$ for which there is a morphism $\beta \in Hom(b,a)_{\mathcal{C}}$ such that $\beta \circ \alpha = \mathrm{id}_a$ and $\alpha \circ \beta = \mathrm{id}_b$.

If we have an isomorphism between objects $a$ and $b$, we call them *isomorphic*.

**Definition 5.3** Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A *functor* $F \colon \mathcal{A} \to \mathcal{B}$ is a procedure that assigns an object $F(a) \in Ob(\mathcal{A})$ to each $a \in Ob(\mathcal{A})$ and a morphism $F(\alpha) \in Hom(F(a), F(b))_{\mathcal{B}}$ to each morphism $\alpha \in Hom(a,b)_{\mathcal{A}}$ such that:

- $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$ for all $a \in Ob(\mathcal{A})$.
- $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$ for all $\alpha \in Hom(a,b)_{\mathcal{A}}$ and $\beta \in Hom(b,c)_{\mathcal{A}}$.

**Example 5.4** The Grothendieck completion defines a functor $G\colon \mathrm{SAb} \to \mathrm{Ab}$. Given an abelian semigroup $V$, $G(V)$ is the Grothendieck completion. Given a semigroup homomorphism $\phi\colon V_1 \to V_2$, we define $G(\phi)\colon G(V_1) \to G(V_2)$ by the formula

$$G(\phi)[x,y] = [\phi(x), \phi(y)] \qquad x, y \in V_1.$$

**Example 5.5** We define the *forgetful functor*, $F\colon \mathrm{Top} \to \mathrm{Set}$ by sending a topological space $X$ to its underlying set, $F(X)$, and a continuous map $f\colon X \to Y$ to the underlying map between the sets $F(X)$ and $F(Y)$.

Let $A$ and $B$ be graded unital $C^*$-algebras with reference supersymmetries $E$ and $F$ respectively. Let $\alpha\colon A \to B$ be a $*$-homomorphism such that $\alpha(E) = F$. Then we have an induced map defined by the formula

$$\alpha_\star(\langle x \rangle_E) = \langle \alpha(x) \rangle_F \qquad x \in SS(A)$$

Similarly, we have maps $\alpha_\star\colon SS_n(A) \to SS_n(B)$ by acting element-wise on matrices.

**Proposition 5.6** *With the above induced maps, $K_1^{(E)}$ is a functor from the category of graded unital $C^\star$-algebras with reference supersymmetry and reference supersymmetry-preserving unital $*$-homomorphisms to the category of abelian groups.*

**Proof:** Let $x \in SS_m(A)$ and $y \in SS_n(A)$. Then

$$\begin{aligned} \alpha_\star(\langle x \rangle_E + \langle y \rangle_E) &= \langle \alpha(x \oplus y) \rangle_F \\ &= \langle \alpha(x) \oplus \alpha(y) \rangle_F \\ &= \alpha_\star(\langle x \rangle_F) + \alpha_\star(\langle y \rangle_F) \end{aligned}$$

So the induced map $\alpha_\star\colon K_1^{(E)}(A) \to K_1^{(F)}(B)$ is a homomorphism of abelian groups.

It is easy to see that if $1\colon A \to A$ is the identity function, then the map $1_\star\colon K_1^{(E)}(A) \to K_1^{(E)}(A)$ is the identity, and that for reference-supersymmetry-preserving unital $*$-homomorphisms $\alpha\colon A \to B$ and $\beta\colon B \to C$, we have the formula $(\beta\alpha)_\star = \beta_\star\alpha_\star$. Hence $K_1^{(E)}$ is a covariant functor, as claimed. □

The following is similar.

**Proposition 5.7** *$K_1$ is a covariant functor.* □

Given a category $\mathcal{C}$, we can form the *opposite category*, $\mathcal{C}^{\mathrm{op}}$. The objects are the same as those of $\mathcal{C}$. We have morphism sets $Hom(a,b)_{\mathcal{C}^{\mathrm{op}}} = Hom(b,a)_{\mathcal{C}}$. For $\alpha \in Hom(a,b)_{\mathcal{C}^{\mathrm{op}}}$ and $\beta \in Hom(b,c)_{\mathcal{C}^{\mathrm{op}}}$, we define composition by

$$\beta * \alpha = \alpha \circ \beta$$

where $\circ$ is the composition in $\mathcal{C}$.

**Definition 5.8** Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A *contravariant functor* $F\colon \mathcal{A} \to \mathcal{B}$ is a functor $F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$.

Thus a contravariant functor is similar to a functor, but reverses the direction of arrows. We sometimes call refer to ordinary (ie: not contravariant) functors as *covariant functors*.

**Example 5.9** Let LCH be the category of locally compact Hausdorff topological spaces and continuous maps. Then we have a contravariant functor $C_0\colon \text{LCH} \to C^*$ defined by sending the space $X$ to the $C^*$-algebra $C_0(X)$.

Given a continuous map $\phi\colon X \to Y$, we define $C_0(\phi) = \phi^*\colon C_0(Y) \to C_0(X)$ by the formula $\phi^*(f) = f \circ \phi$, where $f \in C_0(Y)$.

## 5.2 Homotopy Groups

A *pair*, $(X, A)$, of topological spaces consists of a topological space $X$ along with a subspace $A \subseteq X$. A continuous *map of pairs* $f\colon (X, A) \to (Y, B)$ is a continuous map $f\colon X \to Y$ such that $f[A] \subseteq B$. Given two contrinuous maps of pairs, $f, g\colon (X, A) \to (Y, B)$ a *relative homotopy* from $f$ to $g$ is a continuous map of pairs $H\colon (X \times [0, 1], A \times [0, 1]) \to (Y, B)$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

We call maps of pairs $f, g\colon (X, A) \to (Y, B)$ *relatively homotopic* if there is a relative homotopy between them. Being relatively homotopic is an equivalence relation. We write $[f]$ to denote the relative homotopy class of a map of pairs $f$. We write $[(X, A), (Y, B)]$ to denote the set of all relative homotopy classes of continuous maps of pairs $(X, A) \to (Y, B)$.

**Definition 5.10** Define

$$I^n = [0, 1]^n \qquad \partial I^n = \{(x_1, \dots, x_n) \in [0, 1] \mid x_i = 0 \text{ for some } i\}.$$

Let $X$ be a topological space with basepoint $x_0$. Let $n \in \mathbb{N}$. We define the *n-th homotopy group* of the pair $(X, x_0)$ to be the set of relative homotopy classes

$$\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)].$$

Given two continuous maps of pairs $f, g\colon (I^n, \partial I^n) \to (X, x_0)$, we define the *concatenation* $f * g\colon (I^n, \partial I^n) \to (X, x_0)$ by the formula

$$(f * g)(x_1, \dots, x_n) = \begin{cases} f(2x_1, x_2, \dots, x_n) & x_1 \leq \frac{1}{2} \\ g(2x_1 - 1, x_2, \dots, x_n) & x_1 \geq \frac{1}{2} \end{cases}$$

The following is elementary, and proved similarly to the corresponding result on fundamental groups.

**Proposition 5.11** *The set $\pi_n(X, x_0)$ is a group, with operation defined by the formula $[f][g] = [f * g]$.* □

Another elementary result is the fact that $\pi_n(X, x_0)$ is abelian for $n \geq 2$. Note that the first homotopy group, $\pi_1(X, x_0)$, is the familiar fundamental group, which is not in general abelian.

Note that if $X$ is path-connected, then $\pi_n(X, x_0) \cong \pi_n(X, x_1)$ for all $x_0, x_1 \in X$. When the precise choice of basepoint, $x_0$, is unimportant, we denote the above by $\pi_n(X)$.

Now, let $f\colon (X, x_0) \to (Y, y_0)$ be a basepoint-preserving continuous map. Then we have an induced map $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ defined by the formula $f_*([\gamma]) = [f \circ \gamma]$.

The following is obvious.

**Proposition 5.12** *The correspondence sending a space with basepoint, $(X, x_0)$, to the group $\pi_n(X, x_0)$, and a map $f\colon (X, x_0) \to (Y, y_0)$ to the induced map $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is a functor.*

*Further, if two maps $f, g\colon (X, x_0) \to (Y, y_0)$ are relatively homotopic, then $f_* = g_*$.* □

**Definition 5.13** Let $X$ be a topological space, with basepoint $x_0$. Let $v\colon X \to X$ be the constant map onto $x_0$. We call $X$ *contractible* if $v$ is relatively homotopic to the identity map on $X$.

The following is clear from the above.

**Proposition 5.14** *Let $(X, x_0)$ be contractible. Then the group $\pi_n(X, x_0)$ is trivial for all $n$.* □

We write 0 to denote the trivial group.

Now, consider the *sphere*

$$S^n = \{(x_0, x_1, \dots, x_n) \in R^{n+1} \mid x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}.$$

Pick a basepoint $a \in S^n$. Note that $S^n$ is homeomorphic to the quotient $I^n/\partial I^n$, and we can arrange this homeomorphism so that the equivalence class of the boundary $\partial I^n$ is mapped to the basepoint $a$.

**Proposition 5.15** *The $n$-th homotopy group $\pi_n(X, x_0)$ is in bijective correspondence with the set of relative homotopy classes $[(S^n, a), (X, x_0)]$.*

**Proof:** Let $f\colon (I^n, \partial I^n) \to (X, x_0)$ be contrinuous. Then $f(x) = x_0$ for all $x \in \partial I^n$. Since $S^n$ is homeomorphic to $I^n/\partial I^n$, with $a$ corresponding to $\partial I^n$, we have a unique factorisation

$$f\colon (I^n, \partial I^n) \stackrel{\pi}{\to} (S^n, a) \stackrel{\tilde{f}}{\to} (X, x_0)$$

where $\pi\colon (I^n, \partial I^n) \to (S^n, a)$ comes is the above homeomorphism composed with the quotient map $I^n \to I^n/\partial I^n$. By definition of the quotient topology, the map $\tilde{f}$ is continuous.

We have a map $\alpha\colon [(I^n, \partial I^n), (X, x_0)] \to [(S^n, a), (X, x_0)]$ defined by the formula $\alpha(f) = \tilde{f}$.

Define $\beta\colon [(S^n, a), (X, x_0)] \to [(I^n, \partial I^n), (X, x_0)]$ by the formula $\beta(g) = g \circ \pi$. Then $\beta(\alpha(f)) = f$. By uniquenss of $\tilde{f}$, we have $\alpha(\beta(\tilde{f}) = \tilde{f}$. So $\beta = \alpha^{-1}$, and the map $\alpha$ is a bijection as needed. □

## 5.3 Fibrations

**Definition 5.16** Let $E$ and $X$ be topological spaces. A continuous map $p\colon E \to X$ is called a *fibration*[7] if for any commutative diagram

$$\begin{array}{ccc} I^n & \xrightarrow{g} & E \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ I^{n+1} & \xrightarrow{F} & X \end{array}$$

where $i\colon I^n \to I^{n+1}$ is defined by writine $i(x) = (x, 0)$, there is a map $G\colon I^{n+1} \to E$ fitting into a commutative diagram

$$\begin{array}{ccc} I^n & \xrightarrow{g} & E \\ {\scriptstyle i}\downarrow & \overset{G}{\nearrow} & \downarrow{\scriptstyle p} \\ I^{n+1} & \xrightarrow{F} & X \end{array}$$

**Example 5.17** Let $X$ and $Y$ be topological spaces. Let $p\colon X \times Y \to X$ be the projection map. Suppose we have a commutative diagram

$$\begin{array}{ccc} I^n & \xrightarrow{g} & X \times Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ I^{n+1} & \xrightarrow{F} & X \end{array}$$

Then for $x \in I^n$, we can write $g(x) = (F(x, 0), h(x))$ for some continuous map $h\colon [0, 1] \to Y$. Define $G\colon I^{n+1} \to X \times Y$ by the formula $G(x, t) = (F(x, t), h(x))$. Then $G$ fits into a commutative diagram

$$\begin{array}{ccc} I^n & \xrightarrow{g} & X \times Y \\ {\scriptstyle i}\downarrow & \overset{G}{\nearrow} & \downarrow{\scriptstyle p} \\ I^{n+1} & \xrightarrow{F} & X \end{array}$$

so the projection map $p\colon X \times Y \to X$ is a fibration.

Now, let $X$ be a space with basepoint $x_0$, and let $p\colon E \to X$ be a fibration. We call the inverse image $F = p^{-1}(x_0) \subseteq E$ the *fibre* of $p$.

Pick a basepoint $e_0 \in E$ such that $p(e_0) = x_0$. So $e_0 \in F$.

Suppose we have a continuous map of pairs $\gamma\colon (I^{n+1}, \partial I^{n+1}) \to (X, x_0)$.

Then, as $p$ is a fibration, we have $\tilde{\gamma}\colon I^{n+1} \to E$ such that:

- $p \circ \tilde{\gamma} = \gamma$.
- $ga\tilde{m}ma(x, 0) = e_0$ where $x \in I^n$.

So we have a map of pairs $\tilde{\gamma}(-, 1)\colon (I^n, \partial I^n) \to (F, e_0)$. We can check the following.

[7] Or, more precisely, a *Serre fibration*, though we use no other forms of fibration in these notes.

**Theorem 5.18** *We have a well-defined homomorphism $\partial\colon \pi_{n+1}(E) \to \pi_n(F)$ given by writing $\partial[\gamma] = [\tilde{\gamma}(-,1)]$. This homomorphism fits into a long exact sequence*

$$\to \pi_{n+1}(F) \stackrel{i_*}{\to} \pi_{n+1}(E) \stackrel{p_*}{\to} \pi_{n+1}(X) \stackrel{\partial}{\to} \pi_n(F) \to$$

*where $i\colon F \to E$ is the inclusion.* □

**Definition 5.19** Let $X$ be a metric space, with basepoint $x_0 \in X$. Then we define the *path space* of $X$ to be the metric space

$$PX = \{\gamma \in C([0,1], X) \mid \gamma(0) = x_0\}$$

with metric

$$d(\gamma_1, \gamma_2) = \sup\{d(\gamma_1(t), \gamma_2(t)) \mid t \in [0,1]\}.$$

Actually, we can define a sensible topology- the *compact open topology* on $PX$, defined as above whenever $X$ is a topological space. We do not need to go into details.

**Proposition 5.20** *We have a fibration $p\colon PX \to X$ defined by the formula $p(\gamma) = \gamma(1)$.*

**Proof:** Consider a commutative diagram

$$\begin{array}{ccc} I^n & \stackrel{g}{\longrightarrow} & PX \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ I^{n+1} & \stackrel{F}{\longrightarrow} & X \end{array}$$

So we have, for $x \in I^n$, a map $g(x)\colon [0,1] \to X$ such that $g(x)(1) = F(x,0)$. We want a map $G(x,s)\colon [0,1] \to X$ such that $G(x,0) = g(x)$ and $G(x,s)(1) = F(x,s)$.

Define

$$G(x,s)(t) = \begin{cases} g(x)(s+t) & s+t \leq 1 \\ F(x, s+t-1) & s+t \geq 1 \end{cases}$$

Then this map does what we want. Further, the map $G\colon I^{n+1} \to PX$ is continuous. The result now follows. □

For the above fibration, the fibre is the *loop space*

$$\Omega X = \{\gamma \in C([0,1], X) \mid \gamma(0) = \gamma(1) = x_0\}.$$

The path space $PX$, and so the loop space $\Omega X$ has the constant map, $c$, from $[0,1]$ onto $x_0$ as basepoint.

**Proposition 5.21** *Let $X$ be a metric space, with basepoint $x_0$. Then $\pi_n(\Omega X, c) \cong \pi_{n+1}(X, x_0)$.*

**Proof:** By the above, we have an exact sequence

$$\pi_{n+1}(PX) \to \pi_{n+1}(X) \to \pi_n(\Omega X) \to \pi_n(PX).$$

Let $v\colon PX \to PX$ be the constant map $v(\gamma) = c$ for all $\gamma \in PX$. Define a homotopy $H\colon PX \times [0,1] \to PX$ by the formula $H(\gamma, s)(t) = \gamma(st)$.

Then $H$ is a homotopy between the identity and the constant map $v$. Thus the space $PX$ is contractible, and $\pi_n(PX) = 0$ for all $n$.

Thus the above exact sequence yields an isomorphism $\pi_{n+1}(X) \cong \pi_n(\Omega X)$. □

## 5.4 Direct Limits

**Definition 5.22** Let $\mathcal{C}$ be a category. Let $(a_n)$ be a sequence of objects in $\mathcal{C}$ equipped with morphisms $\lambda_n \in Hom(a_n, a_{n+1})$. Then a *direct limit* of the sequence $(a_n)$ is an object $a \in Ob(\mathcal{C})$ equipped with morphisms $i_n \in Hom(a_n, a)$ such that:

- $i_n = i_{n+1} \circ \lambda_n$ for all $n \in \mathbb{N}$.
- Suppose we have an object $b \in Ob(\mathcal{C})$ equipped with morphisms $j_n \in Hom(a_n, b)$ such that $j_n = j_{n+1} \circ \lambda_n$ for all $n$. Then there is a unique morphism $k \in Hom(a, b)$ such that $k \circ i_n = j$ for all $n$.

The second of the above axioms is called the *universal property* of direct limits. It implies that a direct limit, if it exists, is unique up to isomorphism. We therefore speak of *the* direct limit of a sequence $(a_n)$ rather than just *a* direct limit. We write this direct limit

$$\lim_{\to n} a_n.$$

**Example 5.23** Let $X$ be a topological space. Suppose we have subspaces

$$X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots X$$

such that $X = \cup_{n=1}^{\infty} X_n$.

Let $\lambda_n\colon X_n \to X_{n+1}$ be the inclusion map. Then the sequence $(X_n)$, equipped with these maps, has direct limit $X$ in the category Top. The space $X$ is equipped with the inclusion maps $i_n\colon X_n \to X$.

For our next example, observe that we have inclusions $M_n(\mathbb{C}) \hookrightarrow M_{n+1}(\mathbb{C})$ defined by the formula

$$\left( \begin{array}{ccc} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & vdots \\ x_{n1} & \cdots & x_{nn} \end{array} \right) \mapsto \left( \begin{array}{cccc} x_{11} & \cdots & x_{1n} & 0 \\ \vdots & \ddots & vdots & \vdots \\ x_{n1} & \cdots & x_{nn} & 0 \\ 0 & \cdots & 0 & 0 \end{array} \right) \qquad x_{ij} \in \mathbb{C}$$

or more concisely

$$x \mapsto \left( \begin{array}{cc} x & 0 \\ 0 & 0 \end{array} \right)$$

where $x \in M_n(\mathbb{C})$.

We can therefore form the union

$$M_\infty(\mathbb{C}) = \bigcup_{n=1}^{\infty} M_n(\mathbb{C})$$

The union $M_\infty(\mathbb{C})$ is *not* a pre-$C^*$-algebra, as it is not complete. It is, however, a pre-$C^*$-algebra.

**Definition 5.24** We define the *algebra of compact operators*, $\mathcal{K}$, to be the completion of the pre-$C^*$-algebra $M_\infty(\mathbb{C})$.

**Proposition 5.25** *The sequence $M_n(\mathbb{C})$ has direct limit $\mathcal{K}$.* □

Now, let $A$ be a $C^*$-algebra. We can similarly form the union

$$M_\infty(A) = \bigcup_{n=1}^{\infty} M_n(A)$$

The following result can be deduced from example 2.16.

**Proposition 5.26** *The completion of the pre-$C^*$-algebra $M_\infty(A)$ is the tensor product $A \otimes \mathcal{K}$.* □

It follow that the sequence $(M_n(A))$ has direct limit $A \otimes \mathcal{K}$.

Now, let $H$ be a Hilbert space. We have a $C^*$-algebra, $\mathcal{L}(H)$, consisting of all bounded linear operators on $H$.

An operator $K \in \mathcal{L}(H)$ is called *compact* if it is a norm limit of operators with finite-dimensional images.

It is easy to show that the set $\mathcal{K}(H)$ of compact operators is a closed two-sided ideal. It is therefore a $C^*$-ideal in the $C^*$-algebra $\mathcal{L}(H)$

We thus have a short exact sequence

$$0 \to \mathcal{K}(H) \to \mathcal{L}(H) \to \frac{\mathcal{L}(H)}{\mathcal{K}(H)} \to 0$$

**Definition 5.27** We call the quotient $\mathcal{L}(H)/\mathcal{K}(H)$ the *Calkin algebra*.

The above is related to our earlier $C^*$-algebras in this section by the following.

**Proposition 5.28** *Let $H$ be an infinite-dimensional separable Hilbert space. Then $\mathcal{K}(H) \cong \mathcal{K}$.* □

$\mathcal{K}$ in $C^*$.

## 5.5 Spectra

**Definition 5.29** A *spectrum*, $\mathbb{E}$, is a sequence of topological spaces, $(E_n)$, with basepoints, and contrinuous basepoint-preserving maps $\sigma_n \colon E_n \to \Omega E_{n+1}$.

If these maps are homotopy-equivalences, we call $\mathbb{E}$ an $\Omega$-*spectrum*.

The maps $\sigma_n$ in the above definition are called the *structure maps* of the spectrum $\mathbb{E}$. Note that the structure maps induce maps of homotopy groups

$$\pi_i(E_j) \to \pi_i(\Omega E_{j+1}) \cong \pi_{i+1}(E_{j+1}).$$

**Definition 5.30** We define the *stable homotopy groups* of a spectrum $\mathbb{E}$ to be the direct limits

$$\pi_n(\mathbb{E}) = \lim_{\to k} \pi_{n+k}(E_k).$$

Observe that these are defined for all $n \in \mathbb{Z}$ (not just positive integers), and that if $\mathbb{E}$ is an $\Omega$-spectrum, then $\pi_n(\mathbb{E}) = \pi_{n+k}(E_k)$ whenever $k, n+k > 0$.

**Definition 5.31** Let $\mathbb{E}$ and $\mathbb{F}$ be spectra with structure maps $\sigma_n$ and $\tau_n$ respectively. A *map of spectra* is a sequence of basepoint-preserving continuous maps $f_n \colon E_n \to F_n$ such that $\tau_n \circ f_n = f_{n+1} \circ \sigma_n$ for all $n$.

We can form the category of all spectra and maps of spectra. Given a map of spectra, $f \colon \mathbb{E} \to \mathbb{F}$, there are induced maps $f_* \colon \pi_n(\mathbb{E}) \to \pi_n(\mathbb{F})$. With these induced maps, the stable homotopy groups are functors.

**Definition 5.32** We call a map of spectra $p \colon \mathbb{E} \to \mathbb{X}$ a *level fibration* if each map $p_n \colon E_n \to X_n$ is a fibration of spaces.

The *fibre*, $\mathbb{F}$, is the spectrum where each space $F_n$ is the fibre of the fibration $p_n$, and the structure maps are defined by restricting the structure maps of the spectrum $\mathbb{E}$.

The following now arises from the corresponding result for fibrations of spaces.

**Theorem 5.33** *Let $p \colon \mathbb{E} \to \mathbb{X}$ be a level fibration of spectra, with fibre $\mathbb{F}$ and inclusion $i \colon \mathbb{F} \to \mathbb{E}$. Then we have a long exact sequence of stable homotopy groups*

$$\to \pi_{n+1}(\mathbb{F}) \xrightarrow{i_*} \pi_{n+1}(\mathbb{E}) \xrightarrow{p_*} \pi_{n+1}(\mathbb{X}) \xrightarrow{\partial} \pi_n(\mathbb{F}) \to$$

□

# 6 Elementary Properties of $K$-theory

## 6.1 Stability by Matrix and Clifford Algebras

**Proposition 6.1** *Let $A$ be a unital graded $C^\star$-algebra, equipped with a reference supersymmetry $E$. Then we have an isomorphism*

$$K_1^{(E)}(A) \cong K_1^{(E)}(A \otimes M_n(\mathbb{F}))$$

**Proof:** Let $x \in SS_m(A)$. Let $E^k = E \oplus E \oplus \cdots \oplus E \in SS_k(A)$. Then $x \oplus E^{mn-m} \in SS_{mn}(A) = SS_m(M_n(A))$. It is easy to check that we have a well-defined isomorphism

$$K_1^{(E)}(A) \to K_1^{(E)}(A \otimes M_n(\mathbb{F}))$$

given by the formula

$$\langle x\rangle_E \mapsto \langle x \oplus E^{mn-m}\rangle_{E^n}$$

□

Our next stability result involves considering tensor products with Clifford algebras. To prove it we need the following lemma.

**Lemma 6.2** *Let $A$ be a $C^\star$-algebra equipped with a grading $\epsilon\colon A \to A$. Then the tensor product $A\hat{\otimes}\mathbb{F}_{1,1}$ is naturally isomorphic to the $C^\star$-category $A\hat{\otimes}M_2(\mathbb{F})$ equipped with the non-standard grading*

$$\gamma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \epsilon(a) & -\epsilon(b) \\ -\epsilon(c) & \epsilon(d) \end{pmatrix}$$

*for all morphisms $a, b, c, d \in Hom(A, B)_{\mathcal{A}}$.*

**Proof:** Recall from proposition 3.17 that the Clifford algebra $\mathbb{F}_{1,1}$ is isomorphic to the $C^\star$-algebra $M_2(\mathbb{F})$ equipped with the grading

$$\beta\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} w & -x \\ -y & z \end{pmatrix}$$

The result now follows trivially. □

**Proposition 6.3** *Let $A$ be a unital graded $C^\star$-algebra, equipped with reference supersymmetry $E$. Then we have an isomorphism*

$$K_1^{(E)}(A) \cong K_1^{(E\otimes 1)}(A\hat{\otimes}\mathbb{F}_{1,1})$$

**Proof:** By proposition 6.1 and the above lemma it suffices to prove that there is a natural isomorphism

$$K_1^{(E)}(A\hat{\otimes}M_2(\mathbb{F})) \cong K_1^{(E)}(M_2(A))$$

where $M_2(A)$ denotes the $C^\star$-algebra $A\hat{\otimes}M_2(\mathbb{F})$ equipped with the above grading operator, $\gamma$.

Let us write $\epsilon$ to stand for the usual grading operator on the $C^\star$-algebra $A\hat{\otimes}M_2(\mathbb{F})$. Define

$$V = \begin{pmatrix} 1 & 0 \\ 0 & E_A \end{pmatrix} \in \mathcal{A}\hat{\otimes}M_2(\mathbb{F})$$

Define a $*$-homomorphism $\theta\colon A \otimes M_2(\mathbb{F}) \to A\hat{\otimes}M_2(\mathbb{F})$ by writing $\theta(x) = VxA$ for each $x \in \mathcal{A}\hat{\otimes}M_2(\mathbb{F})$. Then $\theta$ is an isomorphism of (ungraded) $C^\star$-categories.

Observe that $\alpha\theta = \theta\gamma$ and $\theta(E) = E$. Thus the $*$-homomorphism $\theta$ is in fact an isomorphism. The desired result now follows from functoriality of $K_1^{(E)}$. □

Recall that in our definition of $K$-theory, we need to assume that a unital $C^*$-algebra has a supersymmetry. We can drop this assumption.

**Lemma 6.4** *Let $A$ be a unital $C^\star$-algebra. Then we have a reference supersymmetry $E \in A\hat{\otimes}\mathbb{F}_{1,1}\hat{\otimes}M_2(\mathbb{F})$ defined by the matrix*

$$E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

□

We will not prove this lemma now since we will prove a generalisation later on.[8]

**Definition 6.5** Let $A$ be a unital graded $C^\star$-algebra. Let $E$ be the reference supersymmetry in the category $A\hat{\otimes}\mathbb{F}_{1,1}\hat{\otimes}M_2(\mathbb{F})$ defined above.

Then we define $K_1(A)$ to be the abelian group

$$K_1(A) = K_1^{(E)}(A\hat{\otimes}\mathbb{F}_{1,1}\hat{\otimes}M_2(\mathbb{F}))$$

Observe that $K_1$, as defined here, is a functor.

Now if $F \in A$ is an existing reference supersymmetry, we have natural isomorphisms

$$\begin{array}{rcll} K_1^{(F)}(A) & \cong & K_1^{(F\otimes 1)}(A\hat{\otimes}\mathbb{F}_{1,1}) & \text{by proposition 6.3} \\ & \cong & K_1^{(F\otimes 1)}(A\hat{\otimes}\mathbb{F}_{1,1}\hat{\otimes}M_2(\mathbb{F})) & \text{by proposition 6.1} \\ & \cong & K_1^{(E)}(A\hat{\otimes}\mathbb{F}_{1,1}\hat{\otimes}M_2(\mathbb{F})) & \text{by corollary 4.21} \end{array}$$

So our new definition is consistent with the definition made in the original special case. It makes sense to write simply $K_1(A)$ for the $K$-theory group of $A$, whichever equivalent definition we are using.

## 6.2 The Non-Unital Case

In this section we look at how to extend our definitions in order to define $K$-theory for non-unital $C^\star$-algebras.

Recall that for a non-unital $C^*$-algebra $A$, we have a unital $C^\star$-algebra $A^+$ in which $A$ is isometrically embedded. As a vector space (though not in general as a $C^*$-algebra), $A^+$ is the direct sum $A \oplus \mathbb{F}$, and we have a graded unital $\star$-homomorphism $\pi\colon A^+ \to \mathbb{F}$ defined by the formula $\pi(x+\lambda) = \lambda$, where $x \in A$ and $\lambda \in \mathbb{C}$.

The map $\pi$ defines an induced morphism

$$\pi_\star\colon K_1(\mathcal{A}^+) \to K_1(\mathbb{F}_{\mathcal{A}})$$

**Definition 6.6** Let $A$ be a non-unital graded $C^\star$-category. Then we define

$$K_1(A) = \ker \pi_\star$$

Our task is to show that this definition is consistent with the previous definitions made when $A$ is unital. To do this we first look at direct sums.

[8] See proposition **??** in section **??**

**Lemma 6.7** *Let $A$ and $B$ be unital $C^*$-algebras, with reference supersymmetries $E$ and $F$ respectively. Then we have an isomorphism*

$$\phi\colon K_1^{(E,F)}(A \oplus B) \to K_1^{(E)}(A) \oplus K_1^{(F)}(B)$$

*Further, Let $p_A\colon A \oplus B \to A$ and $p_B\colon A \oplus B \to B$ be the obvious projection maps. Then the induced homomorphisms*

$$\begin{array}{l} p_{A\star}\phi^{-1}\colon K_1^{(E)}(A) \oplus K_1^{(F)}(B) \to K_1^{(E)}(A) \\ p_{B\star}\phi^{-1}\colon K_1^{(E)}(A) \oplus K_1^{(F)}(B) \to K_1^{(F)}(B) \end{array}$$

*are also the obvious projection maps.*

**Proof:** Try to define a map $\phi\colon K_1^{(E,F)}(A \oplus B) \to K_1^{(E)}(A) \oplus K_1^{(F)}(B)$ by the formula

$$\phi\langle(x,y)\rangle_{(E.F)} = (\langle x\rangle_E, \langle y\rangle_F)$$

Suppose that $\langle(x,y)\rangle_{(E,F)} = \langle(x',y')\rangle_{(E,F)}$. Then

$$(x,y) \oplus (E^m, F^m) \sim (x',y') \oplus (E^n, F^n)$$

for some $m, n$, meaning, after applying the maps $p_A$ and $p_B$, that

$$x \oplus E^m \sim x' \oplus E^n \qquad y \oplus F^m \sim y' \oplus F^n$$

so $\langle x\rangle_E = \langle x'\rangle_E$, $\langle y\rangle_F = \langle y'\rangle_F$ and the map $\phi$ is well-defined.

By construction it is a homomorphism and is clearly surjective. A similar argument to the above shows injectivity.

Now, consider the induced map $p_{A\star}\colon K_1^{(E,F)}(A \oplus B) \to K_1^{(E)}(A)$.

For any supersymmetry $(x_A, x_B) \in SS_n(A \oplus B)$ we can see that

$$\begin{array}{rcl} p_{A\star}\phi^{-1}(\langle x\rangle_E, \langle y\rangle_F) & = & p_{A\star}\langle(x,y)\rangle_{(E,F)} \\ & = & \langle p_A(x,y)\rangle_E \\ & = & \langle x\rangle_E \end{array}$$

which is the projection map onto the group $K_1^{(E)}(\mathcal{A})$.

The case for $p_B$ is identical. □

We are now ready for the consistency result we were seeking.

**Theorem 6.8** *Let $A$ be a unital graded $C^\star$-algebra, and let $\pi\colon A^+ \to \mathbb{F}$ be the above map associated to the unitisation. Then we have an isomorphism*

$$K_1(A) \cong \mathit{ker}\ \pi_\star$$

.

**Proof:** By proposition 2.30 we have an isomorphism $f\colon A^+ \to A \oplus \mathbb{F}$. By construction, the map $\pi f^{-1}\colon A \oplus \mathbb{F} \to \mathbb{F}$ is the projection map onto $\mathbb{F}$.

So by lemma 6.7 we have a projection map

$$p_\star(f^{-1})_\star\phi^{-1}\colon K_1(A) \oplus K_1(\mathbb{F}_A) \to K_1(\mathbb{F})$$

We therefore have an isomorphism

$$\ker\,(\pi_\star(f^{-1})_\star\phi^{-1}) \cong K_1(A)$$

But the maps $f$ and $\phi$ are isomorphisms. We therefore have anl isomorphism

$$K_1(A) \cong \ker\,\pi_\star$$

and we are are done. □

We have already seen that $K_1$ is a covariant functor from the category of unital graded $C^\star$-algebras to the category of abelian groups. Unsurprisingly, we can extend this to the non-unital case.

**Proposition 6.9** *With induced maps defined by restricting the maps in the unital case, $K_1$ is a functor from the category of all small graded $C^\star$-categories to the category of abelian groups.*

**Proof:** Let $\alpha\colon A \to B$ be a graded $*$-homomorphism. Then we have an induced unital $*$-homomorphism $\alpha^+\colon A^+ \to B^+$, which fits into a commutative diagram

$$\begin{array}{ccc} A^+ & \stackrel{\pi}{\to} & \mathbb{F} \\ \downarrow & & \downarrow \\ B^+ & \stackrel{\pi}{\to} & \mathbb{F} \end{array}$$

Because $K_1$ acts functorially on the category of unital graded $C^\star$-algebras we have another commutative diagram

$$\begin{array}{ccc} K_1A^+ & \stackrel{\pi_\star}{\to} & K_1\mathbb{F} \\ \downarrow & & \downarrow \\ K_1B^+ & \stackrel{\pi_\star}{\to} & K_1\mathbb{F} \end{array}$$

Let $\langle x\rangle_E \in \ker\,(\pi_\star\colon K_1(A^+) \to K_1(\mathbb{F}_\mathcal{A})$. Then it is easy to see that $\alpha_\star\langle x\rangle_E \in \ker\,(\pi_\star K_1(\mathcal{B}^+) \to K_1(\mathbb{F}_\mathcal{B})$. So we have an induced homomorphism $\alpha_\star\colon K_1\mathcal{A} \to K_1\mathcal{B}$ defined by restriction from the unital case, as required. □

Our earlier stability results also extend to the non-unital case.

**Proposition 6.10** *Let $\mathcal{A}$ be a graded $C^\star$-algebra. Then the inclusion*

$$x \mapsto \left(\begin{array}{cc} x & 0 \\ 0 & 0 \end{array}\right)$$

*induces an isomorphism of $K$-theory groups*

$$K_1A \to K_1(A \otimes M_n(\mathbb{F}))$$

**Proof:** We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1(A) & \longrightarrow & K_1(A^+) & \longrightarrow & K_1(\mathbb{F}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_1(A\otimes M_n(\mathbb{F})) & \longrightarrow & K_1(A^+\otimes M_n(\mathbb{F})) & \longrightarrow & K_1(\mathbb{F}\otimes M_n(\mathbb{F})) & \longrightarrow & 0 \end{array}$$

in which the rows are short exact sequences.

By proposition 6.1 the two vertical maps on the right are isomorphisms. Hence the map $K_1(\mathcal{A}) \to K_1(\mathcal{A} \otimes M_n(\mathbb{F}))$ must also be an isomorphism by the 'five lemma' of homological algebra. □

The following result is proved similarly.

**Proposition 6.11** *Let $A$ be any small graded $C^\star$-category. Then we have an isomorphism*

$$K_1^{(E)}(A) \cong K_1^{(E\otimes 1)}(A \otimes \mathbb{F}_{1,1})$$

□

## 6.3 Homotopy-Invariance

**Definition 6.12** Let $A$ and $B$ be graded $C^*$-algebras, and let $\alpha, \beta \colon A \to B$ be graded $*$-homomorphisms.

Then a *homotopy* between $\alpha$ and $\beta$ consists of a continuous map $H \colon A \times [0,1] \to B$ such that

- For each $t \in [0,1]$, the map $H(-,t) \colon A \to B$ is a graded $*$-homomorphism.
- $H(-,0) = \alpha$ and $H(-,1) = \beta$

Suppose that the graded $C^\star$-algebras $A$ and $B$ are unital. Then we call a homotopy $H$ *unital* if $H(1,t) = 1$ for all $t \in [0,1]$.

If a homotopy between graded $*$-homomorphisms $\alpha$ and $\beta$ exists, we say that $\alpha$ and $\beta$ are *homotopic*. It is straightforward to check that the notion of graded $*$-homomorphisms being homotopic is an equivalence relation.

$K$-theory is *homotopy invariant* in the following sense.

**Proposition 6.13** *Let $\alpha, \beta \colon A \to B$ be homotopic graded $*$-homomorphisms. Then the induced maps $\alpha_\star, \beta_\star \colon K_1 A \to K_1 B$ are equal.*

**Proof:** By proposition 6.9 it suffices to prove the result when we have a unital homotopy, $H$, between unital $*$-homomorphisms. By lemma **??** we can further assume that the $C^\star$-algebras $A$ and $B$ have reference supersymmetries $E$ and $F$ respectively, such that $H(E,t) = F$ for each point $t \in [0,1]$.

Consider a point $x \in SS_n(A)$. Then the map $t \mapsto H(x,t)$ is a path in the space $SS_n(B)$ from the point $\alpha(x)$ to the point $\beta(x)$. Hence $\langle \alpha(x) \rangle_F = \langle \beta(x) \rangle_F$, and we are done. □

**Definition 6.14** A graded $C^\star$-algebra $A$ is called *contractible* if the identity map $\mathrm{id}_A \colon A \to A$ and the zero map $0_A \colon A \to A$ are homotopic.

The following is obvious, from the above.

**Proposition 6.15** *If the $C^\star$-category $\mathcal{A}$ is contractible then $K_1\mathcal{A} = 0$* □

**Definition 6.16** A graded $*$-homomorphism $\alpha\colon A \to B$ is called a *homotopy equivalence* if there is a graded $*$-homomorphims $\beta\colon B \to A$ such that the composites $\beta \circ \alpha$ and $\alpha \circ \beta$ are homotopic to the identity maps $1_A$ and $1_B$ respectively.

It is clear from the above that a homotopy-equivalence induces an isomorphism at the level of $K$-theory.

## 6.4 Direct Limits

Suppose we have a sequence $(A_n)$ of graded $C^*$-algebras such that $A_n \subseteq A_{n+1}$ for all $n$. Suppose the union $\bigcup_{n=1}^\infty$ is a dense subset of a graded $C^*$-algebra $A$. It is straightforward to verify that $A$ is the direct limit of the sequence $(A_n)$.

Our main result in this section is that the $K$-theory group $K_1(A)$ is the direct limit of the sequence of $K$-theory groups $K_1(A_n)$. We begin with a lemma.

**Lemma 6.17** *Let $x \in SS(A)$, $\varepsilon > 0$. Then there exists a point $y \in SS(A_j)$ such that*

$$\|\phi_j(y) - x\| < \varepsilon$$

**Proof:** Choose $\varepsilon_1 > 0$. Then there exists an element $a \in A_i$ such that $\|\phi_i(a) - x\| < \varepsilon_1$,

Set

$$b = \frac{a + a^\star}{2}$$

Then the element $b$ is self-adjoint, and since $\phi_i$ is a morphism of $C^\star$-algebras and $x$ is self-adjoint, we can see that $\|\phi_i(b) - x\| < \varepsilon_1$. A similar trick gives us an *odd* self-adjoint element $c \in SS(A_i)$ such that

$$\|\phi_i(c) - x\| < \varepsilon_1$$

Now, without loss of generality, assume that $\varepsilon_1 < \frac{1}{3}$. Observe:

$$\begin{array}{rcl}
\|\phi_i(c)^2 - 1\| & \leq & \|\phi_i(c)^2 - \phi_i(c)x\| + \|\phi_i(c)x - x^2\| \\
& \leq & \|\phi_i(c)\|\|\phi_i(c) - x\| + \|x\|\|\phi_i(c) - x\| \\
& \leq & (\varepsilon_1 + 2\|x\|)\|\phi_i(c) - x\| \\
& \leq & (\varepsilon_1 + 2)\varepsilon_1
\end{array}$$

Define $\varepsilon_2 = \varepsilon_1(2 + \varepsilon_1)$. Then $\varepsilon_1 < \frac{1}{3}$ so $\varepsilon_2 < 1$. Therefore $\|\phi_i(c)^2 - 1\| < 1$. Hence $\|\phi_i(c^2 - 1)\| < 1$.

But the map $\phi_i$ is an inclusion of $C^*$-algebras, and so an isometry. Hence $\|c^2 - 1\| < \varepsilon_2 < 1$.

Since the element $c$ is self-adjoint, we have spectrum

$$Spectrum(c) \subseteq (-1 - \varepsilon_3, -1 + \varepsilon_3) \cup (1 - \varepsilon_3, 1 + \varepsilon_3)$$

where $\varepsilon_3 = \varepsilon_2^{1/2}$. Certainly, $\varepsilon_3 < 1$ so we may define a continuous function $f\colon Spectrum(c) \to \mathbb{R}$ by writing

$$f(t) = \begin{cases} 1 & t \geq 0 \\ -1 & t \leq 0 \end{cases}$$

Hence, by functional calculus, there is a self-adjoint element $y = f(c) \in A_i$ such that $y^2 = 1$. By corollary **??**, the element $y$ is odd since the function $f$ and the element $d$ are odd. So the element $y$ is a supersymmetry.

Now, by the formula expressing the norm of an element of a $C^\star$-algebra in terms of its spectrum, we have

$$\begin{aligned}\|y - c\| &= \|f(d) - d\| \\ &= \sup_{t \in Spectrum(d)} |f(t) - t| \\ &\leq \sup_{t \in (-1-\varepsilon_3, -1+\varepsilon_3) \cup (1-\varepsilon_3, 1+\varepsilon_3)} |f(t) - t| \\ &< \varepsilon_3\end{aligned}$$

To summarise, $y \in SS(A_j)$ and

$$\|\phi_j(y) - x\| \leq \|\phi_j(y - c)\| + \|\phi_j(c) - x\| < \varepsilon_3 + \varepsilon_1$$

Recall that $\varepsilon_3 = (\varepsilon_1(2 + \varepsilon_1))^{1/2}$. Given $\varepsilon > 0$, we can easily find a real number $\varepsilon_1 > 0$ such that $\varepsilon_3 + \varepsilon_1 < \varepsilon$. Hence we have a point $y \in SS(A_j)$ for which $\|\phi_j(y) - x\| < \varepsilon$. □

We now use this lemma to look at paths of supersymmetries.

**Lemma 6.18** *Let $x, y \in SS(A_i)$, and suppose that $\phi_i(x) \sim \phi_i(y)$ in the space $SS(A)$. Then $x \sim y$ in the space $SS(A_j)$ for some $j \geq i$.*

**Proof:** Let $(z_t)$ $(t \in [0,1])$ be a path of supersymmetries in the direct limit $A$ from the supersymmetry $\phi_i(x)$ to the supersymmetry $\phi_i(y)$. For each point $t \in [0,1]$, define an open set

$$U_t = \{z \in SS(A) \mid \|z - z_t\| < 2\}$$

Then, since the interval $[0,1]$ is compact, there exist points $t_0, \ldots, t_n \in [0,1]$ such that $0 = t_0 < t_1 < \cdots < t_n = 1$ and

$$\{z_t \mid t \in [0,1]\} \subseteq \bigcup_{i=0}^{n} U_{t_i}$$

By the above lemma, we may find a point $a_r \in SS(A_{j_r})$ such that $\phi_{j_r}(u_r) \in U_r$ for each $r$. We may also assume that $a_0 = \phi_{ik_0}(x)$ and $a_1 = \phi_{ik_n}(y)$.

Since the collection of open sets $\{U_{t_0}, \ldots, U_{t_n}\}$ covers the path $(z_t)$ it follows that $z_{t_{r+1}} \in U_{t_r}$ for each $r$.

Let $j = \max\{a_0, a_1, \ldots, a_n\}$, so that $a_r \in SS(A_j)$ for all $r$. We have that $\|a_r - a_{r+1}\| < 2$ for all $r$. By proposition 4.4 it follows that $a_r \sim a_{r+1}$ in the space $SS(A_j)$ for each $r$, and so $a_0 \sim a_n$. But, by definition of $A$, $a_0 = x$ and $a_n = y$, so we are done. □

We are now ready for our main result.

**Theorem 6.19** *Suppose we have a sequence $(A_n)$ of graded $C^*$-algebras such that $A_n \subseteq A_{n+1}$ for all $n$. Suppose the union $\bigcup_{n=1}^{\infty}$ is a dense subset of a graded $C^*$-algebra $A$. Then the $K$-theory group $K_1(A)$ is the direct limit of the sequence of $K$-theory groups $K_1(A_n)$.*

**Proof:** By the usual stabilisation tricks we can assume that our $C^*$-algebras are unital, and have non-empty sets of supersymmetries.

Let $x \in SS(A)$. By lemma 6.17, we have $y \in A_i$ such that $\|x - y\| < 2$, so by proposition 4.4, the points $x$ and $y$ lie in the same path-components of the space $SS(A)$. Let $\phi_i \colon A_i \to A$ be the inclusion map. Then the above tells us

$$K_1(A) = \bigcup_{n \in \mathbb{N}} \phi_{i\star} K_1(A_i)$$

Let $\lambda_i \colon A_i \to A_{i+1}$ be the inclusion. Suppose that $G$ is a group equipped with maps $\psi_i \colon K_1(A_i) \to G$ for which $\psi_{i+1}(\lambda_i)_* = \psi_i$ for all $i$.

Then by the above and lemma 6.18 we can define a homomorphism $\beta \colon K_1(A) \to G$ by writing $\beta\phi_{i\star}(x) = \psi_i(x)$ whenever $x \in K_1(A_i)$. Thus the group $K_1()$ has the universal property and so must be the direct limit of the sequence $(K_1(A))$. □

Recall that the $C^*$-algebra of compact operators, $\mathcal{K}$, can be viewed as the closure of the union $\bigcup M_n(\mathbb{C})$ under the inclusions $\lambda_n \colon M_n(\mathbb{C}) \to M_{n+1}(\mathbb{C})$ given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

**Corollary 6.20** *Let $p \in \mathcal{K}$ be a rank one projection. The $*$-homomorphism $A \mapsto A \otimes \mathcal{K}$ defined by writing $a \mapsto a \otimes p$ induces an isomorphism*

$$K_n(\mathcal{A}) \cong K_1(\mathcal{A} \otimes \mathcal{K})$$

*on the level of $K$-theory*

**Proof:** Without loss of generality, say by choosing a suitable basis, we can assume that $p$ is the image in $\mathcal{K}$ of the $1 \times 1$ matrix $(1)$.

By proposition 6.10 each induced map $(\lambda_n)_* \colon K_1(M_k(A)) \to K_1(M_{k+1}(A))$ is an isomorphism. The result now follows by theorem 6.19. □

# 7 Higher $K$-theory

## 7.1 Cones and Suspensions

**Definition 7.1** Let $A$ be a graded $C^\star$-algebra. We define the *cone* and *suspension* of $A$ to be, respectively:

$$CA = \{f \in C[0,1] \otimes A \mid f(0) = 0\} \qquad \Sigma A = \{f \in C[0,1] \otimes A \mid f(0) = f(1) = 0\}$$

These algberas have $C^\star$-algebra structure and grading defined as subalgebras of the tensor product $C[0,1] \otimes \mathcal{A}$.

**Proposition 7.2** *The $C^\star$-algebra $CA$ is contractible*

**Proof:** Define a homotopy $H\colon CA \times [0,1] \to CA$ by the formula

$$H(f,s)(t) = f(st)$$

for all $f \in CA$, and points $s,t \in [0,1]$

Then $H$ is a homotopy between the zero map and identity map on the cone $CA$. □

If we want to have long exact sequences (see later) the only possible definition of the higher $K$-theory groups turns out to be the following.

**Definition 7.3** For each natural number $n \geq 2$ we define inductively:

$$K_{n+1}\mathcal{A} = K_n \Sigma \mathcal{A}$$

Now if $\alpha\colon A \to B$ is any graded $*$-homomorphims, we have an induced graded $*$-homomorphism $\alpha\colon \Sigma A \to \Sigma B$ defined by writing $\alpha(f)(t) = \alpha(f(t))$ for all morphisms $f \in \Sigma A$, $t \in [0,1]$.

With such induced maps, it is clear that taking the suspension is a functor. It follows that each of the $K$-theory groups defines a functor $K_n$. Further, the elementary properties of the functor $K_1$ defined in the previous chapter also work for the functor $K_n$; to be precise, we have the following.

**Proposition 7.4** *If graded $*$-homomorphisms $\alpha, \beta\colon A \to B$ are homotopic, then the induced maps $\alpha_\star$, $\beta_\star\colon K_nA \to K_nB$ are equal.* □

Now, a contractible graded $C^*$-algebra has contractible suspension, so the following holds.

**Corollary 7.5** *Let $A$ be a contractible graded $C^\star$-algebra. Then $K_nA = 0$ for all $n$* □

Similar methods enable us to generalise our earlier stability results.

**Proposition 7.6** *Let $\mathcal{A}$ be a graded $C^\star$-algebra. Then the inclusion*

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$

*induces isomorphisms of $K$-theory groups*

$$K_nA \to K_n(A \otimes M_n(\mathbb{F}))$$

□

**Proposition 7.7** *Let $A$ be a unital graded $C^\star$-algebra with reference supersymmetry, $E$. Then we have a natural isomorphism*

$$K_n^{(E)}(\mathcal{A}) \cong K_n^{(E\otimes 1)}(\mathcal{A} \otimes \mathbb{F}_{1,1})$$

□

**Theorem 7.8** *Let $p \in \mathcal{K}$ be a rank one projection. The $*$-homomorphism $A \mapsto A \otimes \mathcal{K}$ defined by writing $a \mapsto a \otimes p$ induces isomorphisms*

$$K_n(A) \cong K_1(A \otimes \mathcal{K})$$

*on the level of $K$-theory.* □

## 7.2 Spaces of Supersymmetries

Our aim in this section is to produce a space associated to a graded $C^*$-algebra that produces the $K$-theory groups as homotopy groups.

Recall from lemma 6.4 that if $A$ is a unital $C^\star$-algebra we can define a canonical reference supersymmetry

$$E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

in the tensor product $A \otimes \mathbb{F}_{1,1} \otimes M_2(\mathbb{F})$

**Definition 7.9** Let $A$ be a graded $C^\star$-algebra and let $\pi\colon \mathcal{A}^+ \to \mathbb{F}_{\mathcal{A}}$ be the projection map associated to the unitisation. Write $\pi\colon (A^+ \otimes \mathbb{F}_{1,1} \otimes M_2(\mathbb{F})) \to (\mathbb{F} \otimes \mathbb{F}_{1,1} \otimes M_2(\mathbb{F}))_\oplus$ for the obvious induced $*$-homomorphism

Then for each object $n$ we define

$$SS_n^+(A) = \{x \in SS + n(A)_{(A^+\otimes\mathbb{F}_{1,1}\otimes M_2(\mathbb{F}))_\oplus} \mid \pi(x) = E^n \}$$

The obvious induced maps turn $SS_n^+$ into a functor from the category of $C^\star$-algebras to the category of topological spaces with basepoint, $E^n$.

**Proposition 7.10** *Let $B$ be a unital $C^\star$-algebra. Then there is a homeomorphism*

$$SS^+(B) \approx SS(B \otimes \mathbb{F}_{1,1} \otimes M_2(\mathbb{F}))$$

**Proof:**

By proposition **??** there is a commutative diagram

$$\begin{array}{ccc} B^+ & \to & \mathbb{F} \\ \downarrow & & \| \\ B \oplus \mathbb{F} & \to & \mathbb{F} \end{array}$$

in which the horizontal maps are the obvious quotient maps and the vertical map on the left is a natural isomorphism of $C^\star$-algebras.

Forming the tensor product with the matrix $C^\star$-algebra $\mathbb{F}_{1,1} \otimes M_2(\mathbb{F})$ and looking at supersymmetries we obtain a commutative diagram

$$\begin{array}{ccccc} SS^+(B) & \to & SS(B^+ \otimes \mathbb{F}_{1,1} \otimes M_2(\mathbb{F})) & \to & SS(\mathbb{F}_{1,1} \otimes M_2(\mathbb{F})) \\ \downarrow & & \downarrow & & \| \\ SS(B \otimes \mathbb{F}_{1,1} \otimes M_2(\mathbb{F})) & \to & SS((B \oplus \mathbb{F}) \otimes \mathbb{F}_{1,1} \otimes M_2(\mathbb{F})) & \to & SS(\mathbb{F}_{1,1} \otimes M_2(\mathbb{F})) \end{array}$$

in which the space $SS^+(B)$ is the fibre of the map $\pi_\star\colon SS(B^+ \otimes \mathbb{F}_{1,1} \otimes M_2(\mathbb{F})) \to SS(\mathbb{F}_{1,1} \otimes M_2(\mathbb{F}))$ and the central vertical arrow is a homeomorphism.

Thus the map $SS^+(B) \to SS(B \otimes \mathbb{F}_{1,1} \otimes M_2(\mathbb{F}))$ is also a homeomorphism. □

The following is clear.

**Proposition 7.11** *Let $\alpha, \beta\colon A \to B$ and be homotopic $*$-homomorphisms. Then the induced maps $\alpha_\star, \beta_\star\colon SS^+(A) \to SS^+(F(A))$ are relatively homotopic as continuous maps between topological spaces with basepoint.* □

We will now demonstrate how the functor $SS^+$ takes short exact sequences of graded $C^\star$-algebras to fibrations of topological spaces. To do this we need a lemma which says that paths of supersymmetries may be 'lifted' by an epimorphism of graded $C^\star$-algebras.

To prove this lemma we need the following result which converts short paths of supersymmetries into 'easier' paths of elements.

**Proposition 7.12** *Let $\{x_t \mid t \in [0,1]\}$ be a path of supersymmetries in a unital graded $C^\star$-algebra $A$. Then for some $\varepsilon > 0$ there is a path $\{b_t \mid t \in [0,\varepsilon)\}$ of even normal elements of the $C^\star$-algebra $A$ such that*

$$x_t = \exp(-b_t)x_0 \exp(b_t)$$

*for all $t \in [0,\epsilon)$*

**Proof:** Define an element $a_t = (1 + x_0 x_t)/2$. Then the elements $x_0$ and $x_t$ are self-adjoint so we have adjoint $a_t^\star = (1 + x_t x_0)/2$. Hence

$$\begin{aligned} a_t^\star a_t &= (1 + x_t x_0)(1 + x_0 x_t)/4 \\ &= (2 + a_0 a_t + a_t a_0)/4 \qquad \text{since } x_0^2 = x_t^2 = 1 \\ &= a_t a_t^\star \end{aligned}$$

So we have a path $\{a_t \mid t \in [0,1]\}$ of even normal elements of the $C^\star$-algebra $A$. Clearly $a_0 = 1$ so by continuity there exists $\varepsilon > 0$ such that $\|a_t - 1\| < 1$ whenever $t \in [0,\varepsilon)$.

Considering the spectral radius, we see that

$$Spectrum(a_t) \subseteq D(1,1)$$

whenever $t \in [0,\varepsilon)$

There is a holomorphic branch of the complex logarithm function on the disk $D(1,1)$. We can thus apply the functional calculus (see section **??**) to obtain a path $\{b_t \mid t \in [0,\varepsilon)\}$ of even normal elements defined by the formula

$$b_t = \log a_t$$

Now for any point $t \in [0,\varepsilon)$ the spectrum of the element $a_t$ is contained in the disk $D(1,1)$ and so does not contain 0. The element $a_t$ is therefore invertible.

Recall that $a_t = (1 + x_0 x_t)/2$. Hence $x_0 a_t = (x_0 + x_t)/2 = a_t x_t$ and we can write

$$x_t = a_t^{-1} x_0 a_t$$

But by definition of the element $b_t$, $a_t = \exp(b_t)$ and $a_t^{-1} = \exp(-b_t)$. Therefore

$$x_t = \exp(-b_t)x_0 \exp(b_t)$$

whenever $t \in [0,\varepsilon)$ □

We can now prove our lemma.

**Lemma 7.13** *Let $\alpha\colon A \to B$ be a surjective morphism of graded unital $C^\star$-algebras. Let $\{x_t \mid t \in [0,1]\}$ be a path of supersymmetries in $B$.*

*Let $y_0$ be a supersymmetry in $A$ with $\alpha y_0 = x_0$. Then there is a path $\{y_t \mid t \in [0,1]\}$ of supersymmetries in $A$ such that $\alpha y_t = x_t$ for all $t \in [0,1]$*

**Proof:** Let us say that the path $(x_t)$ *lifts* on the interval $[0, s]$ if there is a path $\{z_t \mid t \in [0, s]\}$ of odd involutions in $A$ such that $z_0 = y_0$ and $\alpha z_t = x_t$ for all $t \in [0, s]$

Write

$$S = \sup \{s \in [0,1] \mid (x_t) \text{ lifts on } [0,s]\}$$

Suppose $S < 1$. Then by proposition 7.12 there exists $\varepsilon > 0$ such that there is a continuous path $\{b_t \mid t \in [S, S+\varepsilon)\}$ of even normal elements such that $x_t = \exp(-b_t) x_S \exp(b_t)$ for all $t \in [S, S+\varepsilon)$

Since the morphism $\alpha$ commutes with the grading, by restriction we have a surjective morphism $\alpha|_{A_{\mathrm{even}}} \colon A_{\mathrm{even}} \to B_{\mathrm{even}}$. Hence by corollary 2.37 have a path $\{c_t \mid t \in [S, S+\varepsilon)\}$ of even elements of the $C^\star$-algebra $A$ for which $\alpha c_t = b_t$ whenever $t \in [S, S+\varepsilon)$

For any point $t \in [S, S+\varepsilon)$ define[9]

$$z_t = \exp(-c_t) z_S \exp(c_t)$$

Then we have a path $\{z_t \mid t \in [S, S+\varepsilon)\}$ of odd elements of the $C^\star$-algebra $A$. Further

$$\begin{aligned} (z_t)^2 &= \exp(-c_t) H_S \exp(c_t) \exp(-c_t) z_S \exp(c_t) \\ &= \exp(-c_t)(z_S)^2 \exp(c_t) \\ &= \exp(-c_t) \exp(c_t) \\ &= 1 \end{aligned}$$

so each element $z_t$ is an involution.

Also

$$\begin{aligned} \alpha z_t &= \alpha(\exp(-c_t) z_S \exp(c_t)) \\ &= \exp(-\alpha c_t)(\alpha z_S) \exp(\alpha c_t) \\ &= \exp(-b_t) x_0 \exp(b_t) \\ &= x_t \end{aligned}$$

We are forced to conclude that the path $(z_t)$ lifts on the interval $[0, S+\varepsilon)$. This contradicts the assumption that $S < 1$.

Thus $S = 1$ and we have a path $\{z_t \mid t \in [0,1]\}$ in the space $OI(A)$ such that $\alpha z_t = x_t$ for all $t \in [0,1]$.

By lemma 4.3 the path $(z_t)$ can be deformed to a path $\{y_t \mid t \in [0,1]\}$ of supersymmetries such that $\alpha y_t = x_t$ for all $t \in [0,1]$ and we are done. □

Now, a *short exact sequence* of graded $C^\star$-algebras is an exact sequence of the form

$$0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$$

where the maps $\alpha \colon A \to B$ and $\beta \colon B \to C$ are graded $*$-homomorphisms.

*Exactness* here means that $\alpha$ is injective, $\beta$ is surjective, and $im\, \alpha = \ker \beta$.

We call the following result the *fibration theorem.*

**Theorem 7.14** *The functor $SS^+$ takes a given short exact sequence*

$$0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$$

*of graded $C^\star$-algebras to a fibration*

$$SS^+(A) \overset{\alpha_\star}{\to} SS^+(B) \overset{\beta_\star}{\to} SS^+(C)$$

[9] By the holomorphic functional calculus of section **??**

**Proof:** To begin with let us assume that the $C^\star$-algebras $B$ and $C$ are unital, the morphism $\beta\colon B \to C$ preserves the unit, and the spaces $SS(B)$ and $SS(C)$ are non-empty. The morphism $\beta$ then certainly induces a continuous map $\beta_\star\colon SS(B) \to SS(C)$.

Let $f\colon I^n \to SS(B)$ be a continuous map and let $G\colon I^n \times [0,1] \to SS(C)$ be a continuous map such that $\beta_\star f = G(-,0)$. Then we can consider the map $f$ to be a point in the space $SS(C(I^n) \otimes B)$ and the map $G$ to be a path $(G_t)$ in the space $SS(C(I^n) \otimes C)$.

By theorem 2.36 the morphism

$$\beta_\star\colon C(I^n) \otimes B \to C(I^n) \otimes C$$

is surjective. By lemma 7.13 we can therefore find a path $(F_t)$ in the space $SS(C(I^n) \otimes B)$ such that $F_0 = F$ and $\beta_\star F_t = G_t$. It follows that the map $\beta_\star\colon SS(B) \to SS(C)$ is a fibration.

More generally, if

$$0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$$

is any short exact sequence of $C^\star$-algebras then the induced map

$$\beta_\star\colon SS(B^+ \otimes \mathbb{F}_{1,1} \otimes M_2(\mathbb{F})) \to SS(C^+ \otimes \mathbb{F}_{1,1} \otimes M_2(\mathbb{F}))$$

is a fibration. Considering the quotient maps $\pi_B\colon B^+ \otimes \mathbb{F}_{1,1} \otimes M_2(\mathbb{F}) \to \mathbb{F}_{1,1} \otimes M_2(\mathbb{F})$ and $\pi_C\colon C^+ \otimes \mathbb{F}_{1,1} \otimes M_2(\mathbb{F}) \to \mathbb{F}_{1,1} \otimes M_2(\mathbb{F})$ we have a commutative diagram

$$\begin{array}{ccc} SS(B^+ \otimes \mathbb{F}_{1,1} \otimes M_2(\mathbb{F})) & \to & SS(C^+ \otimes \mathbb{F}_{1,1} \otimes M_2(\mathbb{F})) \\ \downarrow & & \downarrow \\ SS(\mathbb{F}_{1,1} \otimes M_2(\mathbb{F})) & = & SS(\mathbb{F}_{1,1} \otimes M_2(\mathbb{F})) \end{array}$$

where the space $SS^+(B)$ is defined to be the fibre of the vertical map on the left and the space $SS^+(C)$ is defined to be the fibre of the vertical map on the right. Hence the map $\beta_\star\colon SS^+(B) \to SS^+(C)$ defined by restriction is a fibration.

Because the sequence

$$0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$$

is exact the fibre of the map $\beta_\star\colon SS^+(B) \to SS^+(C)$ is simply the image of the map $\alpha_\star\colon SS^+(A) \to SS^+(B)$. But the morphism $\alpha\colon A \to B$ is injective so by proposition **??** the space $SS^+(A)$ is homeomorphic to the image of the map $\alpha_\star$.

We can therefore conclude that the sequence

$$SS^+(A) \overset{\alpha_\star}{\to} SS^+(B) \overset{\beta_\star}{\to} SS^+(C)$$

of topological spaces and continuous maps is a fibration. □

## 7.3 Application to $K$-theory

**Definition 7.15** Let $A$ be a graded $C^*$-algebra. Then we define $SS^+_\infty(A)$ to be the direct limit $\cup_{n\in\mathbb{N}} SS^+_n(A)$.

In this section, we show that $K_n(A) = \pi_n SS_\infty^+(A)$ for all $n \in \mathbb{N}$, and use the fibration theorem to construct long exact sequences of $K$-theory groups.

Note that it is straightforward to see that the assignment $A \mapsto SS_\infty^+(A)$ is a functor, and takes a homotopy between graded $*$-homomorphisms to a homotopy between continuous maps. In particular, if $A$ is contractible, then so is the space $SS_\infty^+(A)$.

**Proposition 7.16** *Let $A$ be a graded $C^*$-algebra. Then there is an isomorphism $K_1(A) \cong \pi_0 SS_\infty^+(A)$.*

**Proof:** Suppose that $A$ is unital, and equipped with a reference supersymmetry, $E$. Let us define a space

$$SS_\infty^{(E)}(\mathcal{F}) = \bigcup_{n=1}^{\infty} SS_n(A)$$

where the inclusion $SS_n(A) \hookrightarrow SS_{n+1}(A)$ is defined by

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & E \end{pmatrix}.$$

We will define an isomorphism $f\colon \pi_0 SS_\infty^{(E)}(\mathcal{F}) \to K_1^{(E)}(\mathcal{A})$; the general result then follows by the usual stabilisation tricks.

Let $\langle x \rangle$ denote the path component of a point $x \in SS_n(A)$ in the space $SS_\infty^{(E)}(A)$. Let $\langle x \rangle_E$ denote the element of the $K$-theory group $K_1^{(E)}(\mathcal{A})$ defined by the supersymmetry $x$ as in definition **??**. Then we can try to define our map $f\colon \pi_0 SS_\infty^{(E)}(\mathcal{F}) \to K_1^{(E)}(\mathcal{A})$ by the formula

$$f(\langle x \rangle) = \langle x \rangle_E$$

Consider points $x_0 \in SS_m(A)$ and $x_1 \in SS_n(A)$ such that $\langle x_0 \rangle = \langle x_1 \rangle$. Then there is a path $\{x_t \mid t \in [0,1]\}$ from the point $x_0$ to the point $x_1$ in the space $SS_\infty^{(E)}(A)$. Such a path is compact, so by proposition **??** there is some $k \geq m, n$, and a path in some space $SS_k(A)$ from the image of the point $x_0$ to the image of the point $x_1$. By definition of the equivalence classes $\langle x_0 \rangle_E$ and $\langle x_1 \rangle_E$ it follows that $\langle x_0 \rangle_E = \langle x_1 \rangle_E$.

The map $f$ is thus well-defined. We would like to show that the map $f$ is bijective. Note that surjectivity is obvious by construction. It remains to show injectivity.

Suppose that $f(\langle x \rangle) = f(\langle y \rangle)$. Then by definition of the equivalence classes $\langle x \rangle_E$ and $\langle y \rangle_E$ there is a path between the images of the supersymmetries $x$ and $y$ in some space $SS_k(A)$. The map $f$ is therefore injective, and we are done □

**Proposition 7.17** *Let*

$$0 \to A \to B \to C \to 0$$

*be a short exact sequence of graded $C^\star$-algebras. Then there is a long exact equence of homotopy groups*

$$\cdots \to \pi_n SS_\infty^+(A) \to \pi_n SS_\infty^+(B) \to \pi_n SS_\infty^+(C) \to \pi_{n-1} SS_\infty^+(A) \to \cdots$$

**Proof:** By theorem 7.14 for all natural numbers $k \in \mathbb{N}$ there is a fibration

$$SS_k^+(A) \to SS_k^+(B) \to SS_k^+(C)$$

and so a corresponding long exact sequence

$$\to \pi_n SS_k^+(A) \to \pi_n SS_k^+(B) \to \pi_n SS_k^+(C) \to \pi_{n-1} SS_k^+(A) \to$$

If we take the direct limit we obtain a sequence

$$\cdots \to \pi_n SS_\infty^+(A) \to \pi_n SS_\infty^+(B) \to \pi_n SS_\infty^+(C) \to \pi_{n-1} SS_\infty^+(A) \to \cdots$$

It is a well-known fact from homological algebra that the direct limit of an exact sequence must be exact. □

**Lemma 7.18** *For any integer $n \geq 0$ there is an isomorphism*

$$\pi_n SS_\infty^+(\Sigma A) \cong \pi_{n+1} SS_\infty^+(A)$$

**Proof:** We have a short exact sequence

$$0 \to \Sigma A \to CA \to A \to 0$$

where $CA$ and $\Sigma A$ are the cone and suspension of $A$ respectively.
By proposition 7.17 there is an exact sequence

$$\pi_{n+1} SS_\infty^+(CA) \to \pi_{n+1} SS_\infty^+(A) \to \pi_n SS_\infty^+(\Sigma A) \to \pi_n SS_\infty^+(CA)$$

Now the cone $CA$ is contractible by proposition 7.2 so the space $SS_\infty^+(C\mathcal{F})$ is contractible. Thus $\pi_{n+1} SS_\infty^+(CA) = 0$ and $\pi_n SS_\infty^+(CA) = 0$. Hence we have an isomorphism

$$\pi_n SS_\infty^+(\Sigma\mathcal{F}) \cong \pi_{n+1} SS_\infty^+(\mathcal{F})$$

and we are done. □

**Proposition 7.19** *Let $A$ be a small graded $C^*$-algebra. Then for any integer $n > 0$ there is an isomorphism*

$$K_n(\mathcal{F}) \cong \pi_{n-1} SS_\infty^+(\mathcal{F})$$

**Proof:** The result holds when $n = 1$ by proposition **??**. Now suppose that there is a natural isomorphism $K_n(\mathcal{F}) \cong \pi_{n-1} SS_\infty^+(\mathcal{F})$ whenever $n \leq N$. Then we have natural isomorphisms

$$\begin{array}{rcll} K_{N+1} & \cong & K_N(\Sigma\mathcal{F}) & \text{by definition} \\ & \cong & \pi_{N-1} SS_\infty^+(\Sigma\mathcal{F}) & \text{by hypothesis} \\ & \cong & \pi_N SS_\infty^+(\mathcal{F}) & \text{by lemma 7.18} \end{array}$$

The desired result thus follows by induction. □

The following important result is immediate.

**Theorem 7.20** *To any short exact sequence*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

*of graded $C^\star$-algebras there is an associated long exact sequence*

$$\to K_nA \to K_nB \to K_nC \to K_{n-1}A \to \cdots \to K_1C$$

*of $K$-theory groups.* □

# 8 Periodicity and Spectra

## 8.1 The Exterior Product

**Definition 8.1** Let $A$ and $B$ be unital graded $C^\star$-algebras. Then we define $A \star B$ to be the $C^\star$-algebra consisting of all continuous maps $\gamma\colon [0, \frac{\pi}{2}] \to A\hat{\otimes}B$ such that $\gamma(0) \in A\hat{\otimes}1$ and $\gamma(\frac{\pi}{2}) \in 1\hat{\otimes}B$.

Given supersymmetries $x \in SS(A), y \in SS(B)$ we define the *exterior product* $x \star y \in SS(A \star B)$ by the formula

$$(x \star y)(\theta) = x \otimes 1 \cos\theta + 1 \otimes y \sin\theta \qquad \theta \in [0, \frac{\pi}{2}]$$

We define $(x \star y)^{-1}$ to be the reversed path

$$(x \star y)^{-1}(\theta) = x \otimes 1 \sin\theta + 1 \otimes y \cos\theta \qquad \theta \in [0, \frac{\pi}{2}]$$

Later on, we will see that we can get by with a simpler construction in the ungraded case.

**Proposition 8.2** *The exterior product is associative.*

**Proof:** Let us define

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^n \mid t_i \geq 0, \sum_{i=0}^{n} t_i^2 = 1\}$$

Then for supersymmetries $x_i \in SS(A_i)$ the exterior product $x_0 \star \cdots \star x_n \in SS(A_1 \star \cdots \star A_n)$ can be identified with a map $\Delta^n \to SS(A \otimes B)$ defined by the formula

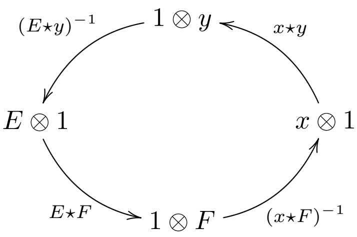
$$(t_0, \ldots, t_n) \mapsto x_0 \otimes 1 \otimes \cdots \otimes 1 t_0 + \cdots + 1 \otimes \cdots \otimes x_n t_n$$

□

**Proposition 8.3** *Suppose that the $C^\star$-algebras $A$ and $B$ are equipped with reference supersymmetries $E$ and $F$ respectively. Then there is an inclusion*

$$i\colon SS(A \star B) \hookrightarrow \Omega SS(A\hat{\otimes}B)$$

**Proof:** Let us write $E$ and $F$ for the reference supersymmetries in the $C^\star$-algebras $A$ and $B$ respectively. Let $1 \otimes F$ be the basepoint of the space of supersymmetries $SS(A \otimes B)$. Then given a path $\gamma \in SS(A \star B)$ from the point $x \otimes 1$ to the point $1 \otimes y$ we define $i(\gamma) \in \Omega SS(A \hat{\otimes} B)$ to be the following composition of paths:




□

The usual stabilisation tricks enable us to extend the above definition to form a map

$$SS^+(A) \times SS^+(B) \to \Omega SS^+(A \hat{\otimes} B)$$

Taking direct limits we obtain a product

$$SS^+_\infty(A) \times SS^+_\infty(B) \to \Omega SS^+_\infty(A \hat{\otimes} B)$$

Looking at homotopy groups, by theorem **??** there is an induced map

$$K_m(A) \otimes K_n(B) \to K_{m+n}(A \hat{\otimes} B)$$

which we write $(\langle x\rangle, \langle y\rangle) \mapsto \langle x\rangle \star \langle y\rangle$.

## 8.2 Bott periodicity

**Definition 8.4** Let $\mathbb{F}_{1,0}$ denote the Clifford algebra generated by one generator, $e$, such that $e^2 = 1$.[10] Let $A$ be a graded $C^*$-algebra. Then we define the *Bott map*

$$b\colon SS^+_\infty(A) \to \Omega SS^+_\infty(A \otimes \mathbb{F}_{1,0})$$

by forming the exterior product $b(x) = x \star -e$.

Recall that a *weak equivalence* is a continuous map $f\colon X \to Y$ such that each induced map $f_*\colon \pi_n(X) \to \pi_n(Y)$ is an isomorphism

The goal of this section is to prove the following major result.

**Theorem** The Bott Periodicity Theorem
The Bott map

$$b\colon SS^+_\infty(A) \to \Omega SS^+_\infty(A \otimes \mathbb{F}_{1,0})$$

is a weak equivalence.

By the usual tricks, it suffices to restrict our attention to a special case.

[10] See section 3.3

**Lemma 8.5** *Suppose that for every graded unital $C^\star$-algebra, $A$, with reference supersymmetry $E$, the Bott map*

$$b\colon SS_\infty^{(E)}(A) \to \Omega SS_\infty^{(1\otimes e)}(A \otimes \mathbb{F}_{1,0})$$

*is a weak equivalence. Then the above result holds.* □

So, let $A$ be a graded $C^\star$-algebra with reference supersymmetry, $E$. Our plan is write the Bott map as a composition of maps

$$SS_\infty^{(E)}(A) \to W_\infty(A) \to \Omega_\infty^{\mathrm{fac}}(A) \to \Omega_\infty(A) \to \Omega SS_\infty^{(1\otimes e)}(A \otimes \mathbb{F}_{1,0})$$

each of which can be shown to be a weak equivalence.

The first step of the proof of the Bott periodicity theorem involves expressing the Bott map in a manner which is amenable to be broken down in this way.

**Lemma 8.6** *For any supersymmetry, $x$, let us write*

$$U_x(\theta) = (\cos\theta 1 \otimes 1 + \sin\theta x \otimes e)$$

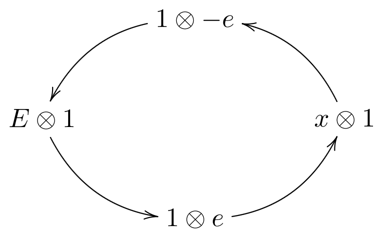
*for $\theta \in [0, \frac{\pi}{2}]$*

*Let $b\colon SS(A) \to \Omega SS(A \otimes \mathbb{F}_{1,0})$ be the Bott map. Then the loop $b(x)$ is homotopic to the loop $\alpha_x$ defined by the formula*

$$\alpha_x(\theta) = U_E(\theta)^\star U_x(\theta)^\star (1 \otimes e) U_x(\theta) U_E(\theta)$$

*and the map $b$ is homotopic to the map $x \mapsto \alpha_x$*

**Proof:** The loop $b(x)$ is the composition of paths




Now

$$U_x(\theta)^\star (1 \otimes e) U_x(\theta) = \cos(2\theta)(1 \otimes e) + \sin(2\theta)(x \otimes 1)$$

so the path $U_x(\theta)^\star(1 \otimes e)U_x(\theta)$ can be identified with the path from the point $1 \otimes e$ to the point $1 \otimes -e$ featuring in the loop $b(x)$ which passes through the point $x \otimes 1$

Similarly the path $U_E(\theta)^\star(1 \otimes -e)U_E(\theta)$ can be identified with the path from the point $1 \otimes -e$ to the point $1 \otimes e$ featuring in the loop $b(x)$ which passes through the point $E \otimes 1$.

But the path obtained by joining the paths $U_x(\theta)^\star(1 \otimes e)U_x(\theta)$ and $U_E(\theta)^\star(1 \otimes -e)U_E(\theta)$ is clearly homotopic to the 'diagonal' path

$$\alpha_x(\theta) = U_E(\theta)^\star U_x(\theta)^\star (1 \otimes e) U_x(\theta) U_E(\theta)$$

and we are done. □

It is in fact the map $x \mapsto \alpha_x$ that we will show to be a weak equivalence.

**Definition 8.7** We define $W_n(A)$ to be the space

$$W_n(A) = \{x \in M_n(A)_{\text{odd}} \mid \textit{Spectrum}(x) \cap i\mathbb{R} = \emptyset\}$$

**Lemma 8.8** *The space $SS_n(A)$ is a strong deformation retraction of the space $W_n(A)$.*

**Proof:** Let us define domains

$$\begin{aligned} D^+ &= \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\} \\ D^- &= \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\} \\ D &= D^+ \cup D^- \end{aligned}$$

We can define a function holomorphic on the domain $D$ by the formula

$$e(z) = \begin{cases} 1 & z \in D^+ \\ 0 & z \in D^- \end{cases}$$

and so by the holomorphic functional calculus[11] define a map $f \colon W_n(A) \to M_n(A)$ by the formula

$$f(x) = 2e(x) - 1$$

Here the element $e(x)$ is defined by writing

$$e(x) = \frac{1}{2\pi i} \int_\gamma \frac{e(z)}{z - x} dz$$

where $\gamma$ is a closed contour in the set $D$ which loops around the spectrum $\textit{Spectrum}(x)$ with winding number 1.

Observe that $f(x)^2 = 1$. Let $\alpha$ be the grading on the $C^\star$-algebra $M_n(A)$. Then since the element $x$ is odd:

$$\alpha(e(x)) = \frac{1}{2\pi i} \int_\gamma \frac{e(z)}{z + x} dz$$

Without loss of generality assume that the contour, $\gamma$, is radially symmetric in the sense that

$$\int_\gamma g(z) dz = - \int_\gamma g(-z) dz$$

for any function $g(z)$ which is holomorphic on a neighbourhood of $\gamma$. Then:

$$\alpha(e(x)) = \frac{-1}{2\pi i} \int_\gamma \frac{e(-z)}{-z + x} dz = \frac{1}{2\pi i} \int_\gamma \frac{1 - e(z)}{z - x} dz = 1 - e(x)$$

Thus $\alpha(f(x)) = -f(x)$ and we have defined a map

$$f \colon W_n(A) \to OI_n(A)$$

into the space of odd involutions of the the $C^\star$-algebra $M_n(A)$.

Suppose that $x \in OI_n(A)$. Then

$$e(x) = \frac{1}{2\pi i} \int_\gamma \frac{e(z)}{z - x} dz = \frac{1}{2\pi i} \int_\gamma \frac{e(z) z}{z^2 - 1} dz + \frac{x}{2\pi i} \int_\gamma \frac{1}{z^2 - 1} dz = \frac{1}{2}(1 + x)$$

[11] See section **??**

by the standard techniques of contour integration. Thus if $x \in OI_n(A)$ then $f(x) = x$.

Now we can define a homotopy between the function $f$ and the identity $1_{W_n(A)}$ by the formula

$$H(x,t) = tx + (1-t)f(x)$$

which proves that the map $f\colon W_n(A) \to OI_n(A)$ is a strong deformation retraction. By lemma 4.3 the space $SS_n(A)$ is thus a strong deformation retraction of the space $W_n(A)$. □

Let us form the direct limit

$$W_\infty(\mathcal{A}) = \varinjlim_n W_n(A)$$

Then by the above lemma the space $SS_\infty^{(E)}(A)$ is a strong deformation retract of the space $W_\infty(A)$ and we are done with the first step of proof of the Bott periodicity theorem.

**Definition 8.9** We write $\Omega_n(A)$ to stand for the space of continuous maps $\gamma\colon [0, \frac{\pi}{2}] \to GL_n(A \otimes \mathbb{F}_{1,0})_{\text{even}}$ such that $\gamma(0) = 1 \otimes 1$ and $\gamma(1) \in GL_n(A)_{\text{even}} \otimes 1$

It is convenient to consider an element, $\gamma$, of the space $\Omega_n(A)$ to be a map from the upper quarter of the unit circle

$$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

to the space $GL_n(A \otimes \mathbb{F}_{1,0})_{\text{even}}$, or equivalently a map defined on the entire unit circle satisfying appropriate symmetry properties. The space $\Omega_n(A)$ is then the space of maps $\gamma\colon S^1 \to GL_n(A \otimes \mathbb{F}_{1,0})_{\text{even}}$ written

$$\gamma(x,y) = f(x,y) \otimes 1 + g(x,y) \otimes e$$

for continuous maps $f$ and $g$ satisfying the formulae

$$\begin{array}{lll} f(1,0) = 1 & g(1,0) = 0 & g(0,1) = 0 \\ f(-x,-y) = f(x,y) & g(-x,-y) = g(x,y) & \\ f(x,-y) = f(x,y) & g(x,-y) = -g(x,y) & \end{array}$$

**Proposition 8.10** *The space $\Omega_n(A)$ is locally convex.*

**Proof:** Let $V$ denote the set of maps $\gamma\colon S^1 \to M_n(A \otimes \mathbb{F}_{1,0})_{\text{even}}$ written

$$\gamma(x,y) = f(x,y) \otimes 1 + g(x,y) \otimes e$$

for which

$$\begin{array}{ll} g(1,0) = 0 & g(0,1) = 0 \\ f(-x,-y) = f(x,y) & g(-x,-y) = g(x,y) \\ f(x,-y) = f(x,y) & g(x,-y) = -g(x,y) \end{array}$$

Then $V$ is a normed vector space. The subset, $S$, of functions $\gamma \in V$ such that $\gamma(x,y) \in GL_n(A \otimes \mathbb{F}_{1,0})_{\text{even}}$ is open and therefore locally convex.

Now the space $\Omega_n(A)$ consists of all functions $\gamma \in S$ for which $\gamma(1,0) = 1 \otimes 1$. Thus the space $\Omega_n(A)$ is also locally convex. □

**Definition 8.11** We write $\Omega_n^{\mathrm{fac}}(A)$ to stand for the space of continuous maps $\gamma\colon S^1 \to GL_n(A \otimes \mathbb{F}_{1,0})_{\mathrm{even}}$ in the space $\Omega_n(A)$ such that we can write

$$\gamma(x,y) = (x1 \otimes 1 + yF \otimes e)(x1 \otimes 1 + yE \otimes e)$$

for some element $F \in W_n(A)$

The following result is easy to see.

**Lemma 8.12** *The map $\chi\colon W_n(A) \to \Omega_n^{\mathrm{fac}}(A)$ defined by the formula*

$$\chi(F)(x,y) = (x1 \otimes 1 + yF \otimes e)(x1 \otimes 1 + yE \otimes e)$$

*is a homeomorphism.* □

As usual we can take direct limits

$$\Omega_\infty(A) = \varinjlim_n \Omega_n(A) \qquad \Omega_\infty^{\mathrm{fac}}(\mathcal{A}) = \varinjlim_n \Omega_n^{\mathrm{fac}}(A)$$

By the above lemma we have a homeomorphism $\chi\colon W_\infty(\mathcal{A}) \to \Omega_\infty^{\mathrm{fac}}(\mathcal{A})$.

Our next lemma, which is the central part of the proof of the Bott periodicity theorem, is far more involved.

**Lemma 8.13** *The inclusion $\Omega_\infty^{\mathrm{fac}}(A) \hookrightarrow \Omega_\infty(A)$ is a weak equivalence.*

The proof of this lemma is simplified by the following result.

**Proposition 8.14** *Suppose that for every $C^\star$-algebra $B$ with reference supersymmetry the induced map $\pi_0\Omega_\infty^{\mathrm{fac}}(B) \to \pi_0\Omega_\infty(B)$ is a bijection. Then the inclusion $\Omega_\infty^{\mathrm{fac}}(A) \hookrightarrow \Omega_\infty(A)$ is a weak equivalence.*

**Proof:** Let $K$ be a compact Hausdorff space. Let $C(K \to X)$ denote the space of continuous functions from the space $K$ into a given space $X$. Observe that $C(K \to \Omega_\infty^{\mathrm{fac}}(A)) = \Omega_\infty^{\mathrm{fac}} C(K \to A)$ and $C(K \to \Omega_\infty(A)) = \Omega_\infty C(K \to A)$.

By hypothesis the inclusion $\Omega_\infty^{\mathrm{fac}} C(K \to A) \hookrightarrow \Omega_\infty C(K \to A)$ induces a bijection $\pi_0\Omega_\infty^{\mathrm{fac}} C(K \to A) \to \pi_0\Omega_\infty C(K \to A)$. Therefore we have an induced bijection

$$\pi_0 C(K \to \Omega_\infty^{\mathrm{fac}}(A)) \to \pi_0 C(K \to \Omega_\infty(A))$$

for every compact Hausdorff space $K$. □

We will now prove lemma 8.13 by expressing the inclusion $\Omega^{\mathrm{fac}}(A) \hookrightarrow \Omega(A)$ as a composite of other inclusions, each of which can be shown to induce a bijection between the relevant sets of path-components.

**Proposition 8.15** *Define $\Omega_n^{\mathrm{pol}}(A)$ to be the space of maps*

$$g(x,y) = f(x,y) \otimes 1 + g(x,y) \otimes e$$

*in the space $\Omega_n(A)$ such that the maps $f(x,y)$ and $g(x,y)$ are polynomials.*

*Then the space $\Omega_n^{\mathrm{pol}}(A)$ is a dense subspace of the space $\Omega_n(A)$.*

**Proof:** Let $P$ denote the space of pairs of functions $(f, g)$ from the circle $S^1$ to the field $\mathbb{F}$ satisfying the conditions

$$\begin{array}{lll} f(1,0) = 1 & g(1,0) = 0 & g(0,1) = 0 \\ f(-x,-y) = f(x,y) & g(-x,-y) = g(x,y) & \\ f(x,-y) = f(x,y) & g(x,-y) = -g(x,y) & \end{array}$$

Let $(f, g) \in \Omega_n(A)$. Then the pair $(f, g)$ is the limit of a sequence of elements of the form

$$x = (f_1, g_1) \otimes a_1 + \cdots + (f_k, g_k) \otimes a_k$$

where $(f_i, g_i) \in P$ and $a_i \in M_n(A \otimes \mathbb{F}_{1,0})_{\text{even}}$.

Let $P^{\text{pol}}$ denote the set of pairs of polynomials which lie in the space $P$. Then by the Stone-Weierstrass theorem[12] the set $P^{\text{pol}}$ is a dense subset of the space $P$.

Let $\varepsilon > 0$. Then we can find a function

$$x = (f_1, g_1) \otimes a_1 + \cdots + (f_k, g_k) \otimes a_k$$

where $(f_i, g_i) \in P$, $a_i \in M_n(A \otimes \mathbb{F}_{1,0})_{\text{even}}$, and $\|x - (f, g)\| < \frac{\varepsilon}{2}$. Since the set $P^{\text{pol}}$ is a dense subset of the space $P$ we can find pairs $(p_i, q_i) \in P^{\text{pol}}$ for which $\|(p_i, q_i) - (f_i, g_i)\| < \frac{\varepsilon}{2k\|a_i\|}$.[13] Hence we have an element

$$y = (p_1, q_1) \otimes a_1 + \cdots + (q_k, q_k) \otimes a_k \in \Omega_n^{\text{pol}}(A)$$

for which $\|y - (f, g)\| < \varepsilon$. Thus the set $\Omega_n^{\text{pol}}(A)$ is a dense subset of the space $\Omega_n(A)$ and we are done. □

The proof of the following result contains a trick that we will use again in the future.

**Proposition 8.16** *The inclusion* $\Omega_n^{\text{pol}}(A) \hookrightarrow \Omega_n(A)$ *induces a bijection* $\pi_0\Omega_n^{\text{pol}}(A) \to \pi_0\Omega_n(A)$

**Proof:** Let $\gamma \in \Omega_n(A)$. The space $\Omega_n(A)$ is locally convex by proposition 8.10 and the set $\Omega_n^{\text{pol}}(A)$ is dense in the space $\Omega_n(A)$ by the above proposition so we can find a point $\gamma' \in \Omega_n^{\text{pol}}(A)$ for which the straight line path joining the points $\gamma$ and $\gamma'$ lies in the space $\Omega_n(A)$. Thus the map $\pi_0\Omega_n^{\text{pol}}(A) \to \pi_0\Omega_n(A)$ induced by the inclusion $\Omega_n^{\text{pol}}(A) \hookrightarrow \Omega_n(A)$ is surjective.

Now consider the $C^\star$-algebra $B = C([0,1] \to A)$. Then $\Omega_n^{\text{pol}}(B) = C([0,1] \to \Omega_n^{\text{pol}}(A))$ and $\Omega_n(B) = C([0,1] \to \Omega_n(A))$. By the above argument the inclusion $\Omega_n^{\text{pol}}(A) \hookrightarrow \Omega_n(A)$ induces a surjection

$$\pi_0 C([0,1] \to \Omega_n^{\text{pol}}(A)) \to \pi_0 C([0,1] \to \Omega_n(A))$$

Hence any two points in the space $\Omega_n^{\text{pol}}(A)$ which are connected by a path in the larger space $\Omega_n(A)$ are also connected by a path in the subspace $\Omega_n^{\text{pol}}(A)$. Thus the induced map $\pi_0\Omega_n^{\text{pol}}(A) \to \pi_0\Omega_n(A)$ is also injective and we are done. □

[12] Any standard textbook on functional analysis, for example [?], may be consulted for details

[13] Without loss of generality we may assume that $a_i \neq 0$.

If we form the direct limit

$$\Omega_\infty^{\mathrm{pol}}(\mathcal{A}) = \varinjlim_n \Omega_n^{\mathrm{pol}}(A)$$

then the map $\Omega_\infty^{\mathrm{pol}}(A) \hookrightarrow \Omega_\infty(A)$ also induces a bijection between the sets of path-components by proposition **??**.

Now let us define $\Omega_n^{\mathrm{quad}}(A)$ to be the space of maps

$$g(x,y) = f(x,y) \otimes 1 + g(x,y) \otimes e$$

in the space $\Omega(A)$ such that the maps $f(x,y)$ and $g(x,y)$ are quadratics.

We have inclusions $\Omega_n^{\mathrm{quad}}(A) \hookrightarrow \Omega_n^{\mathrm{pol}}(A)$. Forming the direct limit we have an inclusion

$$\Omega_\infty^{\mathrm{quad}}(A) \hookrightarrow \Omega_\infty^{\mathrm{pol}}(A)$$

**Proposition 8.17** *The inclusion $\Omega_\infty^{\mathrm{quad}}(A) \hookrightarrow \Omega_\infty^{\mathrm{pol}}(A)$ induces a bijection $\pi_0\Omega_\infty^{\mathrm{quad}}(A) \to \pi_0\Omega_\infty^{\mathrm{pol}}(A)$.*

**Proof:** Let $\gamma \in \Omega_n^{\mathrm{pol}}(A)$. Then the invariance conditions together with the equation $x^2 + y^2 = 1$ which holds for any point $(x,y) \in S^1$ mean that we can write

$$\gamma(x,y) = (1 + c_1 y^2 + \cdots c_m y^{2m}) \otimes 1 + xy(d_1 + d_2 y^2 + \cdots d_m y^{2m-2}) \otimes e$$

where $c_i \in M_n(A)_{\mathrm{even}}$ and $d_j \in M_n(A)_{\mathrm{odd}}$.

Define

$$q(x,y) = \begin{pmatrix} 1 + y^2 c_1 \otimes 1 + xyd_1 \otimes e & y^2 c_2 \otimes 1 + xyd_2 \otimes e & \cdots & y^2 c_m \otimes 1 + xyd_m \otimes e \\ -y^2 & 1 & \cdots & 0 \\ 0 & -y^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \in \Omega_{mn}^{\mathrm{quad}}(A)$$

Define a matrix $N_1$ by writing

$$1 + N_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ y^2 & 1 & \cdots & 0 \\ y^4 & y^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ y^{2m-2} & y^{2m-4} & \cdots & 1 \end{pmatrix}$$

Then $1 + N_1 t \in \Omega_{mn}^{\mathrm{pol}}(A)$ whenever $t \in [0,1]$. Observe that:

$$q(x,y)(1 + N_1) = \begin{pmatrix} \gamma(x,y) & \gamma_2(x,y) & \cdots & \gamma_m(x,y) \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

where the $\gamma_i(x,y)$ are polynomials.

Define a matrix $N_2$ by writing

$$1 + N_2 = \begin{pmatrix} 1 & \gamma_2(x,y) & \cdots & \gamma_m(x,y) \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Then $1 + N_1 t \in \Omega^{\text{pol}}_{mn}(A)$ whenever $t \in [0,1]$. Observe that:

$$q(x,y)(1 + N_1) = (1 + N_2) \begin{pmatrix} \gamma(x,y) & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

Hence we have a path in the space $\Omega^{\text{pol}}_\infty(A)$ from the quadratic $q(x,y)$ to the polynomial $\gamma(x,y)$ defined by the formula

$$(1 + N_2 t)^{-1} q(x,y)(1 + N_1 t)$$

The map $\pi_0 \Omega^{\text{quad}}_\infty(A) \to \pi_0 \Omega^{\text{pol}}_\infty(A)$ induced by the inclusion $\Omega^{\text{quad}}_\infty(A) \hookrightarrow \Omega^{\text{pol}}_\infty(A)$ is therefore surjective. The same trick that was used in proposition 8.16 shows that the map $\pi_0 \Omega^{\text{quad}}_\infty(A) \to \pi_0 \Omega^{\text{pol}}_\infty(A)$ is also injective. □

**Proposition 8.18** *The inclusion* $\Omega^{\text{fac}}_\infty(A) \hookrightarrow \Omega^{\text{quad}}_\infty(A)$ *induces a bijection* $\pi_0 \Omega^{\text{fac}}_\infty(A) \to \pi_0 \Omega^{\text{quad}}_\infty(A)$.

**Proof:** Let $q \in \Omega^{\text{quad}}_n(A)$. Then we can write

$$q(x,y) = x^2 + c \otimes 1 y^2 + d \otimes exy$$

by using the invariance conditions on the space $\Omega^{\text{quad}}_n(A)$ together with the equation $x^2 + y^2 = 1$.

Define a function $r \in \Omega^{\text{quad}}_{2n}(A)$ by writing

$$r(x,y) = \begin{pmatrix} q(x,y) & -E \otimes exy + dE \otimes 1y^2 + cE \otimes exy \\ 0 & 1 \end{pmatrix}$$

where $E$ is the reference supersymmetry in the $C^\star$-algebra $A$.

Observe that we have a linear path of maps in the space $\Omega^{\text{quad}}_{2n}(A)$ from the quadratic $r(x,y)$ to the quadratic $q(x,y) \oplus 1$.

Now we can write

$$r(x,y) = \left( x + \begin{pmatrix} d & E \\ E & 0 \end{pmatrix} \otimes ey \right) \left( x + \begin{pmatrix} 0 & -E \\ -E & 0 \end{pmatrix} \otimes ey \right)$$

Since the point $E$ is a reference supersymmetry there is a path from the quadratic $r(x,y)$ to the quadratic

$$x(x,y) = \left( x + \begin{pmatrix} d & E \\ E & 0 \end{pmatrix} \otimes ey \right) (x + E \otimes ey)$$

which lies in the space $\Omega^{\text{fac}}_\infty(A)$.

The map $\pi_0\Omega_\infty^{\text{fac}}(A) \to \pi_0\Omega_\infty^{\text{quad}}(A)$ induced by the inclusion $\Omega_\infty^{\text{fac}}(A) \hookrightarrow \Omega_\infty^{\text{quad}}(A)$ is therefore surjective. The same trick that was used in proposition 8.16 shows that the map $\pi_0\Omega_\infty^{\text{fac}}(A) \to \pi_0\Omega_\infty^{\text{quad}}(A)$ is also injective. □

**Proof of lemma 8.13:** By proposition 8.16, proposition 8.17, and proposition 8.18 the inclusion $\Omega_\infty^{\text{fac}}(A) \hookrightarrow \Omega_\infty(A)$ induces a bijection $\pi_0\Omega_\infty^{\text{fac}}(A) \to \pi_0\Omega_\infty(A)$.

Hence by proposition 8.14 the inclusion $\Omega_\infty^{\text{fac}}(A) \hookrightarrow \Omega_\infty(A)$ must be a weak equivalence. □

For the last step of the proof of the Bott periodicity theorem, recall the original description of the space $\Omega_n(A)$ as the set of all paths $\gamma\colon [0, \frac{\pi}{2}] \to GL_n(A \otimes \mathbb{F}_{1,0})_{\text{even}}$ such that $\gamma(0) = 1 \otimes e$ and $\gamma(1) \in GL(A)_{\text{even}} \otimes 1$.[14]

**Proposition 8.19** *The quotient map*

$$p\colon GL_n(A \otimes \mathbb{F}_{1,0})_{\text{even}} \to \frac{GL_n(A \otimes \mathbb{F}_{1,0})_{\text{even}}}{GL_n(A)_{\text{even}} \otimes 1}$$

*is a fibration.*

**Proof:** Define an open set

$$\{a \otimes 1 + b \otimes e \in GL(A \otimes \mathbb{F}_{\text{even}} \mid a \in GL(A)_{\text{even}}\}$$

and let $V = p[U]$. Suppose we have an invertible element $x = p^{-1}[V]$. Then $p(x) = p(y)$ for some element $y \in U$. Hence $x = (a \otimes 1)y$ where $a \in GL(A)_{\text{even}}$. Therefore $x \in U$ and we have a homeomorphism

$$f\colon p^{-1}[V] \to (GL(A)_{\text{even}} \otimes 1) \times V$$

defined by the formula

$$f(a \otimes 1 + b \otimes e) = (a \otimes 1, p(a \otimes 1 + b \otimes e))$$

The quotient map

$$p\colon GL_n(A \otimes \mathbb{F}_{1,0})_{\text{even}} \to \frac{GL_n(A \otimes \mathbb{F}_{1,0})_{\text{even}}}{GL_n(A)_{\text{even}} \otimes 1}$$

is therefore a fibre bundle, with fibre $GL(A) \otimes 1$.

Now the topological group $GL(A)_{\text{even}} \otimes 1$ is a closed subgroup of the group $GL(A \otimes \mathbb{F}_{1,0})_{\text{even}}$ so the quotient space $GL_n(A \otimes \mathbb{F}_{1,0})_{\text{even}}/GL_n(A)_{\text{even}} \otimes 1$ is paracompact and Hausdorff. It follows that the quotient map

$$p\colon GL_n(A \otimes \mathbb{F}_{1,0})_{\text{even}} \to \frac{GL_n(A \otimes \mathbb{F}_{1,0})_{\text{even}}}{GL_n(A)_{\text{even}} \otimes 1}$$

is a fibration.[15] □

[14] The fact that this space can also be seen as certain maps on the circle was convenient in the previous stage of the proof.

[15] It is a well-known fact from algebraic topology that a fibre bundle with paracompact and Hausdorff base space is a fibration. See for example section 2.7 of [**?**] for details.

**Corollary 8.20** *The quotient map*

$$p\colon GL_n(A\otimes \mathbb{F}_{1,0})_{\mathrm{even}} \to \frac{GL_n(A\otimes \mathbb{F}_{1,0})_{\mathrm{even}}}{GL_n(A)_{\mathrm{even}}\otimes 1}$$

*induces a weak equivalence*

$$p_\star\colon \Omega_n(A) \to \Omega\left(\frac{GL_n(A\otimes \mathbb{F}_{1,0})_{\mathrm{even}}}{GL_n(A)_{\mathrm{even}}\otimes 1}\right)$$

**Proof:** By proposition 8.19 the induced map

$$p_\star\colon \pi_0\Omega_n(A) \to \pi_0\Omega\left(\frac{GL_n(A\otimes \mathbb{F}_{1,0})_{\mathrm{even}}}{GL_n(A)_{\mathrm{even}}\otimes 1}\right)$$

is a fibration. The fibre is the space of maps $\gamma\colon [0,1]\to GL(A)_{\mathrm{even}}\otimes 1$ such that $\gamma(0)=1\otimes 1$. Hence the fibre is contractible, and the result follows by looking at the long exact sequence of homotopy groups associated to a fibration. □

**Proposition 8.21** *Let us write $X_0$ for the path-component of the basepoint of a topological space, $X$. Then the map*

$$c\colon \left(\frac{GL_n(A\otimes \mathbb{F}_{1,0})_{\mathrm{even}}}{GL_n(A)_{\mathrm{even}}\otimes 1}\right)_0 \to OI_n(A\otimes \mathbb{F}_{1,0})_0$$

*defined by the formula*

$$x\mapsto x^{-1}(1\otimes e)x \text{ for all } x\in GL_n(A\otimes \mathbb{F}_{1,0})_{\mathrm{even}}$$

*is a homeomorphism.*

**Proof:** To begin with, observe that the map $c$ is continuous, well-defined, and open. So all we need to is prove that the map $c$ is bijective.

Let $x,y\in (GL_n(A\otimes \mathbb{F}_{1,0})_{\mathrm{even}})_0$ and suppose that $x^{-1}(1\otimes e)x = y^{-1}(1\otimes e)y$. Then $(1\otimes e)xy^{-1} = xy^{-1}(1\otimes e)$. Let us write $xy^{-1} = a\otimes 1 + b\otimes e$ where $a\in M_n(A)_{\mathrm{even}}$ and $b\in M_n(B)_{\mathrm{odd}}$. Then

$$\begin{array}{rrcl} & (a\otimes 1 + b\otimes e)(1\otimes e) & = & (1\otimes e)(a\otimes 1+b\otimes e)\\ \Rightarrow & a\otimes e + b\otimes 1 & = & a\otimes e - b\otimes 1\\ \Rightarrow & b & = & 0\end{array}$$

Thus if $x^{-1}(1\otimes e)x = y^{-1}(1\otimes e)y$ then $xy^{-1}\in GL_n(A)_{\mathrm{even}}\otimes 1$. It follows that the map $c$ is injective.

Now let us define an equivalence relation on the space $OI_n(A\otimes \mathbb{F}_{1,0})$ by writing $u\sim v$ if $u = x^{-1}vx$ for some element $x\in (GL_n(A\otimes \mathbb{F}_{1,0})_{\mathrm{even}})_0$.

Let $u,v\in OI_n(A\otimes \mathbb{F}_{1,0})$ and suppose that $\|u-v\|<1$. Then we can write $v=u(1+\delta)$ where $\|\delta\|<1$. Observe (by the functional calculus) that:

$$\begin{array}{rrcl} & u(1+\delta)u(1+\delta) & = & 1\\ \Rightarrow & u(1+\delta)u & = & (1+\delta)^{-1}\\ \Rightarrow & (u(1+\delta)^{\frac12}u)^2 & = & (1+\delta)^{-1}\\ \Rightarrow & u(1+\delta)^{\frac12} & = & (1+\delta)^{-\frac12}u\\ \Rightarrow & (1+\delta)^{\frac12}u(1+\delta)(1+\delta)^{-\frac12} & = & u\\ \Rightarrow & (1+\delta)^{\frac12}v(1+\delta)^{-\frac12} & = & u\end{array}$$

Now the element $(1+\delta)^{-\frac{1}{2}}$ belongs to the set $(GL_n(A \otimes \mathbb{F}_{1,0})_{\text{even}})_0$ so we have proved that if $\|u-v\| < 1$ then $u \sim v$. Thus the equivalence classes of the relation $\sim$ are all open subsets of the space $OI_n(A \otimes \mathbb{F}_{1,0})$. Since different equivalence classes are disjoint we have proved that the equivalence classes of the relation $\sim$ are the connected components of the space $OI_n(A \otimes \mathbb{F}_{1,0})$.

By lemma 4.3 and lemma 4.4 the space $OI_n(A \otimes \mathbb{F}_{1,0})$ is locally path-connected, and so components coincide with path-components. Hence the equivalence class of the space $OI_n(A \otimes \mathbb{F}_{1,0})$ containing the point $1 \otimes e$ is simply the path-component, $OI_n(A \otimes \mathbb{F}_{1,0})_0$, containing the point $1 \otimes e$. But the equivalence class containing the point $1 \otimes e$ is by definition the image of the map $c$. Thus the map $c$ is surjective and we are done. □

The above two propositions imply the following lemma.

**Lemma 8.22** *The map $f\colon \Omega_n(A) \to \Omega OI_n(A \otimes \mathbb{F}_{1,0})$ defined by the formula*

$$f(\gamma)(\theta) = \gamma(\theta)^{-1}(1 \otimes e)\gamma(\theta)$$

*is a weak equivalence.* □

Taking direct limits the induced map

$$f\colon \Omega_\infty(A) \to \Omega OI_\infty^{(1\otimes e)}(A \otimes \mathbb{F}_{1,0})$$

is also a weak equivalence.

**Theorem 8.23** *Bott periodicity theorem*
*Let $A$ be a graded $C^*$-algebra. Then the Bott map*

$$b\colon SS_\infty^+(A) \to \Omega SS_\infty^+(A \otimes \mathbb{F}_{1,0})$$

*is a weak equivalence.*

**Proof:** By lemma 8.5, it suffices to prove that the Bott map

$$b\colon SS_\infty^{(E)}(A) \to \Omega SS_\infty^{(1\otimes e)}(A \otimes \mathbb{F}_{1,0})$$

is a weak equivalence whenever $A$ is a graded unital $C^*$-algebra with reference supersymmetry $E$.

By lemma 8.6, in this case the Bott map is homotopic to a composition of maps

$$SS_\infty^{(E)}(A) \to W_\infty(A) \to \Omega_\infty^{\text{fac}}(A) \to \Omega_\infty(A) \to \Omega SS_\infty^{1\otimes e}(A \otimes \mathbb{F}_{1,0})$$

which are all weak equivalences by lemmas 8.8, 8.12, 8.13, and 8.22 respectively.
□

Theorem **??** enables us to relate the Bott periodicity theorem to $K$-theory.

**Corollary 8.24** *Let $\mathcal{A}$ be a small graded $C^\star$-category. Then the Bott map induces natural isomorphisms*

$$b_\star\colon K_r(\mathcal{A}) \cong K_{r+1}(\mathcal{A} \otimes \mathbb{F}_{1,0})$$

*of $K$-theory groups.* □

Finally, the properties of Clifford algebras explored in section 3.3 enable us to show what the Bott periodicity theorem really has to do with periodicity.

**Corollary 8.25** *If $A$ is a complex graded $C^\star$-category then we have natural isomorphisms*

$$K_r(\mathcal{A}) \cong K_{r+2}(\mathcal{A})$$

**Proof:** We have natural isomorphisms

$$\begin{array}{rcll} K_r(\mathcal{A}) & \simeq & K_{r+2}(\mathcal{A} \otimes \mathbb{C}_{1,0} \otimes \mathbb{C}_{1,0}) & \text{by the above} \\ & \simeq & K_{r+2}(\mathcal{A} \otimes \mathbb{C}_{2,0}) & \text{by proposition 3.16} \\ & \simeq & K_{r+2}(\mathcal{A} \otimes \mathbb{C}_{1,1}) & \text{by corollary 3.19} \\ & \simeq & K_{r+2}(\mathcal{A}) & \text{by proposition 6.3} \end{array}$$

□

Similarly, we have the following.

**Corollary 8.26** *If $\mathcal{A}$ is a real graded $C^\star$-category then we have natural isomorphisms*

$$K_r(\mathcal{A}) \cong K_{r+8}(\mathcal{A})$$

□

## 8.3 $K$-theory Spectra

**Definition 8.27** Let $A$ be a graded $C^\star$-algebra. Then we define $\mathbb{K}A$ to be the spectrum with spaces

$$(\mathbb{K}A)_n = SS^+_\infty(A \otimes \mathbb{F}_{n+1,0})$$

and structure maps

$$b\colon SS^+_\infty(A \otimes \mathbb{F}_{n,0}) \to \Omega SS^+_\infty(A \otimes \mathbb{F}_{n+1,0})$$

The spectrum $\mathbb{K}A$ is called the *$K$-theory spectrum* of the graded $C^\star$-algebra $A$.

Note that the structure maps in the $K$-theory spectrum are all weak equivalences. A spectrum with this property is termed an *$\Omega$-spectrum.*

**Proposition 8.28** *Let $\mathcal{A}$ be any small graded $C^\star$-category. Then*

$$K_r\mathcal{A} = \pi_r\mathbb{K}\mathcal{A} \text{ for all } r \in \mathbb{Z}$$

**Proof:** Let $r \geq 0$. Then:

$$\begin{array}{rcll} \pi_r\mathbb{K}A & = & \pi_r(\mathbb{K}A)_0 & \text{since } \mathbb{K}A \text{ is an } \Omega\text{-spectrum} \\ & = & \pi_r SS^+_\infty(A \otimes \mathbb{F}_{1,0}) & \\ & = & K_r A & \text{by theorem ?? and Bott periodicity} \end{array}$$

If $r < 0$ note that the induced maps

$$b_\star\colon \pi_{r+n}(\mathbb{K}A)_n \to \pi_{r+n+1}(\mathbb{K}A)_{n+1}$$

are still well-defined isomorphisms for all sufficiently large $n$.

Choose such a number $n$ for which $n \equiv 0 \bmod 8$. Then:

$$\begin{aligned} \pi_r \mathbb{K}A &= \pi_{r+n}(\mathbb{K}A)_n \\ &= K_{r+n}(A \otimes \mathbb{F}_{n+1,0}) \\ &= K_r A \qquad \text{by Bott periodicity} \end{aligned}$$

and we are done. □

# 9 Computations

## 9.1 $K_0$ of an ungraded $C^*$-algebra

**Definition 9.1** Let $A$ be a $C^*$-algebra. An element $p \in A$ is called a *projection* if $p = p^\star$ and $p^2 = p$.

We write $P(A)$ for the set of projections in $A$, and $P_n(A) = P(M_n(A))$.

**Lemma 9.2** *Let $A$ be unital. Then there is a natural homeomorphism $f\colon P(A) \to SS(A \otimes \mathbb{F}_{1,0})$ for which $f(0) = 1 \otimes e$.*

**Proof:** We can define such a homeomorphism by the formula

$$f(p) = (1 - 2p) \otimes e$$

□

**Definition 9.3** Let $p \in P_m(A)$ and $q \in P_n(A)$ be projections. Let us write $0_k$ to denote the $k \times k$ zero matrix. Then we write $p \sim_h q$ when the projections $p \oplus 0_{n+k}$ and $q \oplus 0_{m+k}$ lie in the same path-component of the space $P_{m+n+k}(A)$ for some $k$. We write $[p]$ for the equivalence class containing the projection $p$.

Form the set $V_0(A)$ of all equivalence classes of projections in matrices over $A$. Then the set $V_0(A)$ is an abelian semigroup with group operation induced by taking the direct sum of projections.

**Theorem 9.4** *The $K$-theory group $K_0(A)$ is naturally isomorphic to the Grothendieck completion of the semigroup $V_0(A)$.*

**Proof:** By the Bott periodicity theorem we can define

$$K_0(A) = K_1(A \otimes \mathbb{F}_{1,0})$$

Thus, following definition 4.13, the group $K_0(A)$ is the set of formal differences

$$\{\langle x\rangle - \langle y\rangle \mid x, y \in SS_n(A \otimes \mathbb{F}_{1,0})\}$$

which by lemma 9.2 is naturally isomorphic to the group of formal differences

$$\{\langle p\rangle - \langle q\rangle \mid p, q \in P_n(A)\}$$

Let $G$ be the Grothendieck completion of the semigroup $V_0(A)$. Then we can define a homomorphism $\alpha\colon K_0(A) \to G$ by the formula

$$\alpha(\langle p\rangle - \langle q\rangle) = [p] - [q]$$

Suppose that $p \in P_m(A)$ and $q \in P_n(A)$ are projections. Then

$$[p] - [q] = [p \oplus 0] - [0 \oplus q] = \alpha(\langle p \oplus 0_B\rangle - \langle 0 \oplus q\rangle)$$

so the homomorphism $\alpha$ is surjective.

Suppose that $p, q \in P_n(A)$ and $[p] = [q]$. Then for some $k$ we know that the projections $p \oplus 0_k$ and $q \oplus 0_k$ lie in the same path-component of the space $P_{n+k}(A)$. Hence

$$\begin{array}{rcl} \langle p \oplus 0\rangle & = & \langle q \oplus 0\rangle \\ \Rightarrow \quad \langle p\rangle + \langle 0\rangle & = & \\ langleq\rangle + \langle 0\rangle & & \\ \Rightarrow \quad \langle p\rangle & = & \langle q\rangle \end{array}$$

Thus the homomorphism $\alpha$ is also injective and we are done. □

An alternative equivalence relation between projections is sometimes useful.

**Definition 9.5** Projections $p \in P_m(A)$ and $q \in P_n(A)$ are said to be *unitarily equivalent* if we can find a unitary element $u \in Hom(A, B)$ such that $p = u^\star q u$.

**Proposition 9.6** *Let $p \in P(A)$ and $q \in P(B)$ be unitarily equivalent. Then $p \sim_h q$.*

**Proof:** Let $p = u^\star q u$ where $u \in Hom(A, B)$ is unitary. Then we can define a unitary matrix

$$v = \begin{pmatrix} 0 & u^\star \\ u & 0 \end{pmatrix} \in Hom(A \oplus B, A \oplus B)$$

such that $p \oplus 0 = v^\star(0 \oplus q)v$.

By lemma **??** there is a path of unitary matrices, $(v_t)$, from the matrix $\begin{pmatrix} v & 0 \\ 0 & v^\star \end{pmatrix}$ to the identity.

Hence we have a path of projections, $(v_t(0 \oplus q \oplus 0)v_t^\star)$, from the projection $p \oplus 0 \oplus 0$ to the projection $0 \oplus q \oplus 0$. Therefore $p \sim_h q$. □

The following definition and results can prove useful when we need 'more room' than is present in the space $P(A)$.

**Definition 9.7** Let $\mathcal{A}$ be a unital $C^\star$-category and let $A \in Ob(\mathcal{B})$. Then a morphism $p \in Hom(A, A)$ is called an *idempotent* if $p^2 = p$.

We write $I(A)$ for the set of idempotents based at the object $A$. Observe that any projection is also an idempotent.

**Lemma 9.8** *The space $P(A)$ is a strong deformation retraction of the space $I(A)$.*

**Proof:** The map

$$p \mapsto (1-2p) \otimes e$$

from lemma 9.2 gives us homeomorphisms

$$P(A) \to SS(A \otimes \mathbb{F}_{1,0}) \qquad I(A) \to OI(A \otimes \mathbb{F}_{1,0})$$

But by lemma 4.3 the space of odd involutions $OI(A \otimes \mathbb{F}_{1,0})$ is a strong deformation retract of the space of supersymmetries $SS(A \otimes \mathbb{F}_{1,0})$. □

**Proposition 9.9** *Suppose we have projections $p \in P(A)$ and $q \in P(B)$ together with an invertible element $x \in Hom(A, B)$ for which $p = x^{-1}qx$. Then $p \sim_h q$*

**Proof:** Form the matrix

$$y = \begin{pmatrix} 0 & x^{-1} \\ x & 0 \end{pmatrix} \in GL(A \oplus B)$$

Then $p \oplus 0 = y^{-1}(0 \oplus q)y$. By proposition **??** there is a strong deformation retraction $f\colon GL(A \oplus B) \to U(A \oplus B)$. There is thus a path of idempotents between the projection $f(y)^\star(0 \oplus q)f(y)$ and the projection $p \oplus 0 = y^{-1}(0 \oplus q)y$.

By the above lemma it follows that $p \oplus 0 \sim_h f(y)^\star(0 \oplus q)f(y)$. But the projections $f(y)^\star(0 \oplus q)f(y)$ and $0 \oplus q$ are unitarily equivalent so by proposition 9.6 we have the relation $p \sim_h q$ and we are done. □

Let $\mathcal{A}$ and $\mathcal{B}$ be small graded $C^\star$-categories. Recall from section 8.1 that we can define a natural product

$$SS^+_\infty(\mathcal{A}) \times SS^+_\infty(\mathcal{B}) \to \Omega SS^+_\infty(\mathcal{A} \otimes \mathcal{B})$$

by sending a pair $(x, y)$ of supersymmetries to a loop of supersymmetries $x \star y$. At the level of $K$-theory groups we have an induced product

$$K_m(\mathcal{A}) \otimes K_n(\mathcal{B}) \to K_{m+n}(\mathcal{A} \otimes \mathcal{B})$$

for natural numbers $m, n \geq 1$.

By the Bott periodicity theorem we can define the $K$-theory group $K_0(\mathcal{A})$ to be equal to the group $K_1(\mathcal{A} \otimes \mathbb{F}_{1,0})$ for any small $C^\star$-category $\mathcal{A}$.

## 9.2 $K_1$ of an ungraded $C^*$-algebra

## 9.3 Examples of $K_0$ and $K_1$

# 10 Topological $K$-theory

## 10.1 Vector Bundles

# References