Planar Transformations

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January 31, 2013

1 Translation, Scaling, Reflection

Matrix techniques are used in computer graphics. Many programs need to manipulate a two-dimensional picture, for example by moving it, changing its size, or rotating it. Here we show how to describe these notions mathematically, as linear transformations. This is convenient as we can then use matrix methods.

Translation

Translation of a picture by distance a horizontally and distance b vertically (ie: by (a, b)) moves a point with coordinates (x, y) to a point with coordinates (x', y'), where

$$x' = x + a \qquad y' = y + b.$$

Scaling

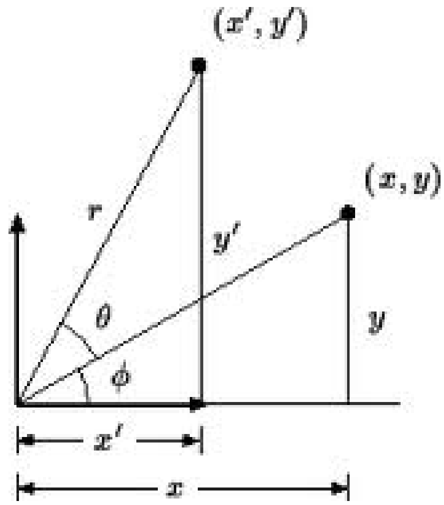
Let $\alpha > 0$. Scaling by a factor of α moves a point with coordinates (x, y) to a point with coordinates (x', y'), where

$$x' = \alpha x \qquad y' = \alpha y.$$

This scaling increases size if $\alpha > 1$, and decreases size if $\alpha < 1$.

Rotation

To describe rotation in terms of mathematical formulae, we need some geometry. Suppose we are rotating anticlockwise about the origin by an angle θ. Consider a point (x, y), and write $x = R\cos\phi$, $y = R\sin\phi$. Let the rotation move it to point (x', y').



Then, as in the picture,

$$x' = R\cos(\phi + \theta) \qquad y' = R\sin(\phi + \theta).$$

Now, using the double angle formulae for cos and sin, and substituting back in the formulae for x and y in terms of R and ϕ, we see

$$x' = R\cos\phi\cos\theta - R\sin\phi\sin\theta = x\cos\theta - y\sin\theta$$

and

$$y' = R\cos\phi\sin\theta \ + R\sin\phi\cos\theta = x\sin\theta + y\cos\theta.$$

2 Matrix Form

Let us write our coordinates as column vectors $\begin{pmatrix} x \\ y \end{pmatrix}$. Then scaling by a factor α is a linear transformation

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}$$

which can be represented by the matrix

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}.$$

Rotating anticlockwise about the origin by an angle θ is a linear transformation

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

which can be represented by the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Unfortunately, translation, as written, is not a linear transformation, and cannot be represented by a matrix. However, we can cheat. The cheat is to represent a point (x, y) by the vector

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

Observe

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+a \\ y+b \\ 1 \end{pmatrix}.$$

So, if we write coordinates as vectors in this way, translation by (a, b) is represented by the matrix

$$T_{a,b} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, scaling by α and rotation by θ are represented by the matrices

$$S_\alpha = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

respectively.

3 Composing Transformations

We can use the above to represent composite transformations, consisting of doing more than one of the above to a picture.

Remember, if we have linear transformations S and T, with matrices A and B respectively, and we apply T then S to a vector v, we get a vector

$$S(T(v)) = S \circ T(v) = ABv.$$

That is to say the transformation coming from doing T, then S, has matrix AB. Notice that the order in which we write down the matrices is the *opposite* to the order in which we do the transformations. This is an inevitable feature of our mathematical notation, which is annoying but no big deal. In any case, we can't ignore it.

Example

What is the matrix of the transformation that comes from translating by $(2, -1)$ and then rotating by 90 degrees anticlockwise ?

Solution:

The translation matrix is

$$T = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

The rotation matrix is

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So the resulting transformation has matrix (don't forget to get the correct order):

$$RT = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$