MAS165: Solutions to Chapter 5 Problems

Question 1

In spherical polar coordinates (r, θ, ϕ). the sphere is described by

$$0 \le r \le a \quad 0 \le \theta \le \pi \quad 0 \le \phi \le 2\pi$$

and

$$dV = r^2 \sin\theta \; dr \; d\theta \; d\phi$$

so the total charge is

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \frac{q_0 r^3}{a^3} r^2 \sin\theta \; dr \; d\theta \; d\phi = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \frac{q_0 r^5 \sin\theta}{} dr \; d\theta \; d\phi$$

$$= \int_0^{2\pi} \frac{q_0 a^6 \sin\theta}{6a^3} \; d\theta \; d\phi = \frac{2\pi q_0 a^3}{6} \int_0^{\pi} \sin\theta \; d\theta = \frac{2\pi q_0 a^3}{3}$$

Question 2

1. The curve C is parametrised by

   $$\boldsymbol{r}(t) = (t, t^2, 2t^2 - t) \qquad 0 \le t \le 1$$

   Here

   $$\boldsymbol{F}(\boldsymbol{r}(t)) = (2t - t^2, t, t) \qquad \frac{d\boldsymbol{r}}{dt} = (1, 2t, 4t - 1)$$

   so

   $$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 (2t - t^2, t, t) \cdot (1, 2t, 4t - 1) \; dt$$

   $$= \int_0^1 2t - t^2 + 2t^2 + 4t^2 - t \; dt = \int_0^1 5t^2 + t \; dt = \frac{5}{3} + \frac{1}{2} = \frac{13}{6}.$$

2. The curve C is parametrised by

   $$\boldsymbol{r}(t) = (t, t, t) \qquad 0 \le t \le 1$$

   Here

   $$\boldsymbol{F}(\boldsymbol{r}(t)) = (t, t, t) \qquad \frac{d\boldsymbol{r}}{dt} = (1, 1, 1)$$

so

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 (t, t, t) \cdot (1, 1, 1)\ dt$$
$$= \int_0^1 3t\ dt = \frac{3}{2}.$$

If $\boldsymbol{F}$ were irrotational, the path integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ would not depend on C, only on the endpoints of C. But we have just found two paths with the same endpoints where the above integral is different. So $\boldsymbol{F}$ is *not* irrotational, ie: $\boldsymbol{F}$ is rotational.

Question 3

Let $L = \sqrt{1+x^3}$ and $M = 2xy$, so $\boldsymbol{F} = (L, M)$ and

$$\frac{\partial M}{\partial x} = 2y \qquad \frac{\partial L}{\partial y} = 0.$$

By Green's theorem

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int\int_A \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} dx\ dy = \int\int_A 2y\ dx\ dy$$

Now the line from $(0, 0)$ to $(1, 3)$ is given by $y = 3x$, so the area A is described by

$$0 \leq y \leq 3x \qquad 0 \leq x \leq 1$$

and

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 \int_0^{3x} 2y\ dy\ dx = \int_0^1 \left[y^2\right]_0^{3x}\ dx = \int_0^1 9x^2 = \left[3x^3\right]_0^1 = 3$$

Question 4

In cylindrical polar coordinates, C has the parametrisation $r = R$, $z = 0$ and $\theta = t$, where $0 \leq t \leq 2\pi$. As a vector, $\boldsymbol{r} = R\hat{\boldsymbol{r}}$. So

$$\boldsymbol{H}(\boldsymbol{r}(t)) = \frac{H_0 R^2}{a^2}\hat{\boldsymbol{\theta}}$$

and

$$\frac{d\boldsymbol{r}}{dt} = R\frac{d\hat{\boldsymbol{r}}}{dt} = R\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} = R\hat{\boldsymbol{\theta}}$$

so

$$\oint_C \boldsymbol{H} \cdot d\boldsymbol{r} = \int_0^{2\pi} \frac{H_0 R^2}{a^2}\hat{\boldsymbol{\theta}} \cdot R\hat{\boldsymbol{\theta}}\ dt = \frac{2\pi H_0 R^3}{a^2}$$

Question 5

The curve C is parametrised by

$$\boldsymbol{r}(t) = (-1+4t, 2-2t, t) \qquad 0 \le t \le 1$$

Here

$$\boldsymbol{F}(\boldsymbol{r}(t)) = (-t+4t^2, 0, 2t^2-2t) \qquad \frac{d\boldsymbol{r}}{dt} = (4, -2, 1)$$

so

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 (-t+4t^2, 0, 2t^2-2t) \cdot (4,-2,1)\ dt$$
$$= \int_0^1 18t^2 - 6t\ dt = 6 - 3 = 3.$$

Question 6

The surface S is parametrised by

$$\boldsymbol{r}(u,v) = (u, v, 0) \qquad 0 \le u \le 1,\ 0 \le v \le 1$$

Here

$$\boldsymbol{F}(\boldsymbol{r}(u,v)) = (v, u, u+v)$$

and

$$\boldsymbol{n}\ dS = \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\ du\ dv = (1,0,0) \times (0,1,0)\ du\ dv = (0,0,1)\ du\ dv$$

So

$$\int\int_S \boldsymbol{F} \cdot \boldsymbol{n}\ dS = \int_0^1 \int_0^1 (v, u, u+v) \cdot (0,0,1)\ du\ dv = \int_0^1 \int_0^1 u+v\ du\ dv$$
$$= \int_0^1 \frac{1}{2} + v\ dv = \frac{1}{2} + \frac{1}{2} = 1.$$

Question 7

We have parametrisation

$$\boldsymbol{r}(\theta, \phi) = a_1 \hat{\boldsymbol{r}} \qquad 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi$$

and

$$\boldsymbol{E}(\boldsymbol{r}(\theta,\phi)) = \frac{E_0 a_1^2}{a^2} \hat{\boldsymbol{r}} \qquad \boldsymbol{n}\ dS = a_1^2 \sin\theta\ \hat{\boldsymbol{r}}\ d\theta\ d\phi$$

so

$$\int\int_S \boldsymbol{E} \cdot \boldsymbol{n}\ dS = \int_0^{2\pi} \int_0^{\pi} \frac{E_0 a_1^2}{a^2} a_1^2 \sin\theta\ d\theta\ d\phi = 2\pi \int_0^{\pi} \frac{E_0 a_1^4}{a^2} \sin\theta\ d\theta = \frac{4\pi E_0 a_1^4}{a^2}$$

Question 8

Let R be the interior of the cylinder, where $0 \le x^2 + y^2 \le 4$ and $0 \le z \le 2$. Then by Gauss' divergence theorem

$$\int\int_S \boldsymbol{F} \cdot \boldsymbol{n}\; dS = \int\int\int_R \boldsymbol{\nabla} \cdot \boldsymbol{F}\; dV$$

Now in cylindrical polar coordinates, R is given by

$$0 \le r \le 2 \qquad 0 \le \theta \le 2\pi \qquad 0 \le z \le 2$$

and

$$dV = r\; dr\; d\theta\; dz$$

We have

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = y^2 + 3y^2 + y^2 = 5y^2 = 5r^2 \sin^2\theta$$

so

$$\int\int\int_R \boldsymbol{\nabla} \cdot \boldsymbol{F}\; dV = \int_0^2 \int_0^{2\pi} \int_0^2 5r^3 \sin^2\theta\; dr\; d\theta\; dz$$

$$= 2 \times \frac{1}{2} \times 2\pi \times \left[\frac{5}{4} r^4\right]_0^2 = 40\pi$$

Question 9

Let R be the interior of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \ge 0$. Then by Gauss' divergence theorem

$$\int\int_S \boldsymbol{F} \cdot \boldsymbol{n}\; dS = \int\int\int_R \boldsymbol{\nabla} \cdot \boldsymbol{F}\; dV$$

Now in spherical polar coordinates (r, θ, ϕ), R is given by $0 \le r \le a$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le 2\pi$, and we have

$$dV = r^2 \sin\theta\; dr\; d\theta\; d\phi$$

We have

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = 2z + z + 2z = 5z = 5r\cos\theta$$

so

$$\int\int\int_R \boldsymbol{\nabla} \cdot \boldsymbol{F}\; dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a 5r^3 \sin\theta \cos\theta\; dr\; d\theta\; d\phi$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \frac{5}{4} a^4 \sin\theta\cos\theta\; d\theta\; d\phi = \int_0^{2\pi} \frac{5}{4} a^4 \left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2} d\phi = 2\pi \times \frac{5}{8} a^4 = \frac{5\pi a^4}{4}$$

Question 10

For $x < a$, we have

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{3E_0x^2}{a^3}$$

For $x > a$, we have $\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0$. In the region τ

$$\iiint_\tau \boldsymbol{\nabla}\cdot\boldsymbol{E}\,dV = \int_{-a_1}^{a_1}\int_{-b}^{b}\int_{-c}^{c}\frac{3E_0x^2}{a^3}\,dx\,dy\,dz = 4bc\left[\frac{E_0x^3}{a^3}\right]_{-a_1}^{a_1} = \frac{8E_0a_1^3bc}{a^3}.$$

The surface $\partial\tau$ of the region τ consists of:

- Two planes $x = \pm a_1$, with $-b \le y \le b$ and $-c \le z \le c$. Here

$$\boldsymbol{n}\,dS = \pm\boldsymbol{i}\,dy\,dz$$

- Two planes where $y = \pm b$ and

$$\boldsymbol{n}\,dS = \pm\boldsymbol{j}\,dx\,dz$$

- Two planes where $z = \pm c$ and

$$\boldsymbol{n}\,dS = \pm\boldsymbol{k}\,dx\,dz$$

We have

$$vE = \frac{E_0x^3}{a^3}\boldsymbol{i}$$

so $\boldsymbol{E}\cdot\boldsymbol{n}\,dS = 0$ on the last two sets of planes. Hence

$$\iint_{\partial\tau}\boldsymbol{E}\cdot\boldsymbol{n}\,dS = \int_{-c}^{c}\int_{-b}^{b}E_0\frac{a_1^3}{a^3}\boldsymbol{i}\cdot\boldsymbol{i}\,dy\,dz - \int_{-c}^{c}\int_{-b}^{b}E_0\frac{(-a_1)^3}{a^3}\boldsymbol{i}\cdot\boldsymbol{i}\,dy\,dz = \frac{8E_0a_1^3bc}{a^3}.$$

Question 11

In cylindrical polar coordinates we have

$$\boldsymbol{J} = \boldsymbol{\nabla}\times\boldsymbol{H} = \frac{1}{r}\begin{vmatrix}\hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & \hat{\boldsymbol{z}}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ 0 & H_0r^3/a^2 & 0\end{vmatrix} = \frac{3H_0r^2}{a^2}\hat{\boldsymbol{z}}$$

The curve C is parametrised by

$$\boldsymbol{r}(\theta) = R\hat{\boldsymbol{r}} \qquad 0 \le \theta \le 2\pi$$

Here

$$\boldsymbol{H}(\boldsymbol{r}(\theta)) = H_0\left(\frac{R}{a}\right)^2\hat{\boldsymbol{\theta}} \qquad \frac{d\boldsymbol{r}}{d\theta} = R\hat{\boldsymbol{\theta}}$$

so

$$\oint_C \boldsymbol{H} \cdot d\boldsymbol{r} = \int_0^{2\pi} \frac{H_0 R^3}{a^2}\ d\theta = \frac{2\pi H_0 R^3}{a^2}.$$

On the other hand, on the plane surface S, in cylindrical polar coordinates we have $0 \leq r \leq R$, $0 \leq \theta \leq 2\pi$ and $z = 0$. Further

$$\boldsymbol{n} dS = \hat{\boldsymbol{z}}\ r\ dr\ d\theta$$

so

$$\int\int_S \boldsymbol{J} \cdot \boldsymbol{n}\ dS = \int_0^{2\pi} \int_0^R \frac{3H_0 r^2}{a^2}\ dr d\theta = \frac{2\pi H_0 R^3}{a^2}.$$