MAS165: Solutions to Chapter 4 Problems

Question 1

The region $\mathcal{R}$ is given by $0 \leq y \leq 1$ and $0 \leq x \leq 1-y$ so

$$\int\int_{\mathcal{R}} x + 2y \ dx \ dy = \int_0^1 \int_0^{1-y} x + 2y \ dx \ dy = \int_0^1 \left[\frac{1}{2}x^2 + 2xy\right]_0^{1-y} dx \ dy$$

$$= \int_0^1 \frac{1}{2}(1-y)^2 + 2y(1-y) \ dy = \int_0^1 \frac{1}{2} + y - \frac{3}{2}y^2 \ dy = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \frac{1}{2}$$

Question 2

The integral is

$$\int_0^1 dy \int_{2y}^2 wx{+}y \ dx = \int_0^1 \int_{2y}^2 2x{+}y \ dx \ dy = \int_0^1 \left[x^2 + xy\right]_{2y}^2 \ dy = \int_0^1 4{+}2y{-}6y^2 \ dy$$

$$= \left[4y^2 + y^2 - 2y^3\right]_0^1 = 3$$

This integral is

$$\int\int_R 2x + y \ dx \ dy$$

where R is the region

$$2y \leq x \leq 2 \qquad 0 \leq y \leq 1$$

Drawing it, R is the triangle between the lines $y = 0$, $x = 1$, and $y = \frac{1}{2}x$, that is

$$0 \leq x \leq 2 \qquad 0 \leq y \leq \frac{1}{2}x$$

So

$$\int\int_R 2x + y \ dx \ dy = \int_0^2 \int_0^{\frac{1}{2}} x2x + y \ dy \ dx = \int_0^2 \left[2xy + \frac{1}{2}y^2\right]_0^{x/2} \ dx$$

$$= \int_0^2 x^2 + \frac{1}{8}x^2 \ dx = \left[\frac{3}{8}x^3\right]_0^2 = 3$$

Question 3

The region $\mathcal{R}$ is defined by $y^2 \le x \le 1$ and $0 \le y \le 1$ (draw it!), so

$$\int\int_{\mathcal{R}} 2x + y \; dx \; dy = \int_0^1 \int_{y^2}^1 2x + y \; dx \; dy = \int_0^1 \left[x^2 + xy\right]_{x=y^2}^{x=1} \; dy$$

$$= \int_0^1 1 + y - y^3 - y^4 \; dy 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} = \frac{21}{20}$$

Question 4

In plane polar coordinates (r, θ), the region $\mathcal{R}$ is defined by $0 \le r \le a$ and $0 \le \theta \le \frac{\pi}{2}$. Now

$$(x^2 + y^2)^{\frac{1}{2}} = r \qquad dx \; dy = r \; dr \; d\theta$$

so

$$\int\int_{\mathcal{R}} = \int_0^{\frac{\pi}{2}} \int_0^a r^2 \; dr \; d\theta = \frac{\pi}{2} \times \frac{1}{3} a^3 = \frac{\pi a^3}{6}$$

Question 5

Let $\mathcal{R}$ be the region $0 \le r \le a$, $0 \le \theta \le 2\pi$ in plane polar coordinates (r, θ). Let

$$\sigma = \frac{\sigma_0 r^2}{a^2}$$

Then the mass is

$$M = \int\int_{\mathcal{R}} \sigma \; dx \; dy = \int_0^{2\pi} \int_0^a \frac{\sigma_0 r^2}{a^2} r \; dr \; d\theta = \frac{2\pi\sigma_0}{a^2} \int_0^a r^3 \; dr = \frac{\pi \sigma_0 a^2}{2}$$

Question 6

Let R be the region of the (x, y)-plane where $x^2 + y^2 \le 16$. Then

$$V = \int\int_R x^2 + y^2 \; dx \; dy$$

In plane polar coordinates (r, θ)

$$x^2 + y^2 = r^2 \qquad dx \; dy = r \; dr \; d\theta$$

and R is given by

$$0 \le r \le 4 \qquad 0 \le \theta \le 2\pi$$

so

$$V = \int_0^{2\pi} \int_0^4 r^2 r \; dr \; d\theta = \int_0^{2\pi} \left[\frac{1}{4} r^4\right]_0^4 d\theta = \int_0^{2\pi} 64 d\theta = 128\pi$$

Question 7

We want

$$M = \int\int_R \sigma_0 \; dx \; dy$$

where R is defined by

$$a \leq y \leq 2a \qquad 0 \leq x \leq \sqrt{4a^2 - y^2}$$

ie:

$$M = \sigma_0 \int_a^{2a} \int_0^{\sqrt{4a^2-y^2}} \; dx \; dy = \sigma_0 \int_a^{2a} \sqrt{4a^2 - y^2} \; dy$$

Now by the hint

$$M = \left[\frac{y}{2} \sqrt{4a^2 - y^2} + 2a^2 \sin^{-1} \left(\frac{y}{2a} \right) \right]_a^{2a} = \sigma_0 (2\pi a^2 - a\sqrt{3a^2} - \frac{2\pi a^2}{3}) = \frac{\sigma_0 a^2 (4\pi - 3\sqrt{3})}{3}$$