MAS165: Solutions to Chapter 3 Problems

Question 1

(i) $\boldsymbol{\nabla} f = (yz, xz, xy)$.

(ii) $\boldsymbol{\nabla} f = (y+z, x+z, x+y)$.

(iii) $\boldsymbol{\nabla} f = (6x, 0, -8z)$.

(iv) $\boldsymbol{\nabla} f = (-e^{-x}\sin y, e^{-x}\cos y, 0)$.

Question 2

We have

$$\boldsymbol{\nabla}\phi = (3x^2 + z, z, x + y)$$

A unit vector in the direction of $\boldsymbol{r}$ is

$$\hat{\boldsymbol{r}} = \frac{1}{\sqrt{5}}(1, 2, 0).$$

So we have directional derivative

$$\boldsymbol{\nabla}\phi(1,2,3)\cdot\hat{\boldsymbol{r}} = \boldsymbol{\nabla}\phi(1,2,3)\cdot\frac{1}{\sqrt{5}}(1,2,0) = \frac{1}{\sqrt{5}}(6,3,3)\cdot(1,2,0) = \frac{12}{\sqrt{5}}$$

Question 3

We have the formula *curl grad* $f = \mathbf{0}$ for all f, so in all parts of this question, *curl* $\boldsymbol{F} = \mathbf{0}$.

(i)

$$\boldsymbol{F} = (yz, xy, xy) \qquad \mathit{div}\,\boldsymbol{F} = 0$$

(ii)

$$\boldsymbol{F} = (y+z, x+z, x+y) \qquad \mathit{div}\,\boldsymbol{F} = 0$$

(iii)

$$\boldsymbol{F} = (6x, 0, -8z) \qquad \mathit{div}\,\boldsymbol{F} = 6 - 8 = -2$$

(iv)

$$\boldsymbol{F} = (-e^{-x}\sin y, e^{-x}\cos y, 0) \qquad \mathit{div}\,\boldsymbol{F} = e^{-x}\sin y - e^{-x}\sin y = 0$$

Question 4

(i) Let $\phi(x,y,z) = x^2 + y^2 + z^2$, so the surface is $\phi(x,y,z) = 4$. Observe

$$\boldsymbol{\nabla}\phi = (2x, 2y, 2z).$$

At $(1,1,\sqrt{2})$:

$$\boldsymbol{\nabla}\phi(1,1,\sqrt{2}) = (2,2,2\sqrt{2}) \qquad |\boldsymbol{\nabla}\phi(1,1,\sqrt{2}) = \sqrt{4+4+8} = 4$$

so we have unit normal

$$\boldsymbol{n} = \frac{1}{4}(2,2,2\sqrt{2}) = \left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$$

Hence the plane has equation

$$\boldsymbol{r} \cdot \boldsymbol{n} = (1,1,\sqrt{2}) \cdot \boldsymbol{n} = \frac{1}{2} + \frac{1}{2} + 1 = 2$$

(ii) Let $\phi(x,y,z) = x^2 - y^2 - z^2$, so the surface is $\phi(x,y,z) = 0$. Observe

$$\boldsymbol{\nabla}\phi = (2x, -2y, -2z).$$

At $(1,1,0)$:

$$\boldsymbol{\nabla}\phi(1,1,0) = (2,-2,0) \qquad |\boldsymbol{\nabla}\phi(2,-2,0) = \sqrt{4+4} = 2\sqrt{2}$$

so we have unit normal

$$\boldsymbol{n} = \frac{1}{\sqrt{2}}(1,-1,0)$$

Hence the plane has equation

$$\boldsymbol{r} \cdot \boldsymbol{n} = (1,1,0) \cdot \boldsymbol{n} = 0$$

Question 5

The angle between surfaces is the angle between the normal vectors to their tangent planes.

(i) Let

$$\phi_1(x,y,z) = 3x^2 + 2y^2 - z \qquad \phi_2(x,y,z) = 6x - y^2 - z = 0$$

Then the surfaves are given by $\phi_1(x,y,z) = 0$ and $\phi_2(x,y,z) = 0$. Now

$$\boldsymbol{\nabla}\phi_1 = (2x, 4y, -1) \qquad \boldsymbol{\nabla}\phi_2 = (6, -2y, -1)$$

so we have normal vectors

$$\boldsymbol{n_1} = \boldsymbol{\nabla}\phi_1(1,1,5) = (6,4,-1) \qquad \boldsymbol{n_2} = \boldsymbol{\nabla}\phi_2(1,1,5) = (6,-2,-1)$$

The angle, θ, between $\boldsymbol{n_1}$ and $\boldsymbol{n_2}$ is given by

$$\cos\theta = \frac{\boldsymbol{n_1}\cdot\boldsymbol{n_2}}{|\boldsymbol{n_1}||\boldsymbol{n_2}|} = \frac{36-8+1}{\sqrt{53}\sqrt{41}} = 0.622$$

so

$$\theta = 0.899$$

(51.5 degrees)

(ii) **Note: The second surface here should be $x^2 + y^2 = 8$.**
Let

$$\phi_1(x,y,z) = x^2 + y^2 - z^2 \qquad \phi_2(x,y,z) = x^2 + y^2$$

Then the surfaces are given by $\phi_1(x,y,z) = 0$ and $\phi_2(x,y,z) = 8$. Now

$$\boldsymbol{\nabla}\phi_1 = (2x, 2y, -2z) \qquad \boldsymbol{\nabla}\phi_2 = (2x, 2y, 0)$$

so we have normal vectors

$$\boldsymbol{n_1} = \boldsymbol{\nabla}\phi_1(2,2,\sqrt{8}) = (4,4,-4\sqrt{2}) \qquad \boldsymbol{n_2} = \boldsymbol{\nabla}\phi_2(2,2,\sqrt{8}) = (4,4,0)$$

The angle, θ, between $\boldsymbol{n_1}$ and $\boldsymbol{n_2}$ is given by

$$\cos\theta = \frac{\boldsymbol{n_1}\cdot\boldsymbol{n_2}}{|\boldsymbol{n_1}||\boldsymbol{n_2}|} = \frac{32}{8\times 4\sqrt{2}} = \frac{1}{\sqrt{2}}$$

so

$$\theta = \frac{\pi}{4}$$

(90 degrees)

Question 6

We have $V = \ln(x^2 + y^2)$, so

$$\boldsymbol{\nabla}V = \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}, 0\right) \qquad \boldsymbol{E} = \left(\frac{-2x}{x^2+y^2}, \frac{-2y}{x^2+y^2}, 0\right)$$

Now

$$\boldsymbol{\nabla}^2 V = \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} + \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2} = \frac{4}{x^2+y^2} - \frac{4(x^2+y^2)}{(x^2+y^2)^2} = 0$$

Question 7

Subce $\boldsymbol{H}$ is continuous when $y = \pm a$, and constant for $y > a$, we have

$$\boldsymbol{H}(x, y, z) = \boldsymbol{H}(x, a, z) = H_0 \boldsymbol{k}$$

Similarly, for $y < -a$, we have

$$\boldsymbol{H} = -H_0 \boldsymbol{k}$$

For $|y| < a$:

$$\boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{H} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_0(y^3/a^3) \end{vmatrix} = H_0 \left(\frac{3y^2}{a^3} \right) \boldsymbol{i}$$

Since $\boldsymbol{H}$ is constant, for $|y| > a$, we have $\boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{0}$.

Question 8

We have

$$\boldsymbol{\nabla}\phi = (2x - 3y, -3x + 4y, 0) \qquad \boldsymbol{\nabla}\phi(0, 1, 1) = (-3, 4, 0)$$

Thus a unit vector which is in the direction of the maximum rate of change is $(-\frac{3}{5}, \frac{4}{5}, 0)$. The rate of change has magnitude

$$|v\nabla\phi(0, 1, 1)| = 5.$$

Question 9

We have $\boldsymbol{E} = E(x)\boldsymbol{i}$, so $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{dE}{dx}$. Let $-a \leq x \leq a$. Then

$$dEdx = 5\rho_0 x^4 \qquad E(0) = 0$$

Hence $E(x) = \rho_0 x^5$. If $|x| > a$, then $\rho = 0$, so $\frac{dE}{dx} = 0$ and E is constant. We see

$$E(x) = \begin{cases} -\rho_0 a^5 & x < -a \\ \rho_0 x^5 & -a \leq x \leq a \\ \rho_0 a^5 & x > a \end{cases}$$

Question 10

In spherical polar coordinates (r, θ, ϕ)

$$\boldsymbol{\nabla} V = \frac{\partial V}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\phi}}$$

If there is no dependence on θ and ϕ, then the partial derivatives with respect to these variables are zero, and

$$\nabla\phi = \frac{dV}{dr}\hat{\boldsymbol{r}} = \frac{1}{r}\frac{dV}{dr}\boldsymbol{r}$$

For $\phi = e^{-r}/r^2$, we have

$$\nabla\phi = \frac{-r^2e^{-r} - 2re^{-r}}{r^5}\boldsymbol{r} = -e^{-r}\left(\frac{1}{r^3} + \frac{2}{r^4}\right)\boldsymbol{r}$$

Question 11

We have

$$V = \left(r - \frac{a^3}{r^2}\right)\sin\theta\sin\phi$$

and

$$\nabla V = \frac{\partial V}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\boldsymbol{\phi}}$$

so

$$\nabla V = \left(1 + \frac{2a^3}{r^3}\right)\sin\theta\sin\phi\hat{\boldsymbol{r}} + \left(1 - \frac{a^3}{r^3}\right)\cos\theta\sin\phi\hat{\boldsymbol{\theta}} + \left(1 - \frac{a^3}{r^3}\right)\cos\phi\hat{\boldsymbol{\phi}}$$

Question 12

For $r < a$:

$$\boldsymbol{J} = \nabla\times\boldsymbol{H} = \frac{1}{r}\begin{vmatrix} \hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{z} \\ 0 & H_0r^3/a^3 & 0 \end{vmatrix} = \frac{3H_0r}{a^2}\hat{\boldsymbol{z}}$$

For $r > a$:

$$\boldsymbol{J} = \nabla\times\boldsymbol{H} = \frac{1}{r}\begin{vmatrix} \hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{z} \\ 0 & H_0a & 0 \end{vmatrix} = \boldsymbol{0}$$

Question 13

We have

$$\nabla\phi = (ze^x\cos y, -ze^x\sin y, e^x\cos y)$$

A unit vector in the direction of $\boldsymbol{i} + 2\boldsymbol{j}$ is

$$\boldsymbol{u} = \frac{1}{\sqrt{5}}(1, 2, 0).$$

So we have directional derivative

$$\nabla\phi(1, 0, \pi/3)\cdot\boldsymbol{u} = (\frac{\pi e}{3}, 0, e)\cdot\frac{1}{\sqrt{5}}(1, 2, 0) = \frac{\pi e}{3\sqrt{5}}$$