MAS165: Solutions to Chapter 2 Problems

Question 1

- $$\frac{d(f\boldsymbol{A})}{dt} = \frac{df}{dt}\boldsymbol{A}+f\frac{d\boldsymbol{A}}{dt} = 2t(t,e^t,-t^2)+(t^2-3)(1,e^t,-2t) = (3t^2-3,(t^2+2t-3)e^t,-4t^3+6t)$$

- $$\frac{d(\boldsymbol{B}\cdot\boldsymbol{C})}{dt} = \frac{d\boldsymbol{B}}{dt}\cdot\boldsymbol{C}+\boldsymbol{B}\cdot\frac{d\boldsymbol{C}}{dt} = (-2\sin 2t, 2\cos 2t, 0)\cdot(t,t^2,-3t)+(\cos 2t,\sin 2t,0)\cdot(1,2t,-3)$$
  $$= (2t^2+1)\cos 2t$$

- We use
  $$\frac{d(\boldsymbol{B}\times\boldsymbol{C})}{dt} = \frac{d\boldsymbol{B}}{dt}\times\boldsymbol{C}+\boldsymbol{B}\times\frac{d\boldsymbol{C}}{dt}$$
  and
  $$\frac{d\boldsymbol{B}}{dt}\times\boldsymbol{C} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2\sin 2t & 2\cos 2t & 0 \\ t & t^2 & -3t \end{vmatrix} = -6t\cos 2t\boldsymbol{i}-6t\sin 2t\boldsymbol{j}+(-2t^2\sin 2t-2t\cos 2t)\boldsymbol{k}$$
  $$\boldsymbol{B}\times\frac{d\boldsymbol{C}}{dt} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos 2t & \sin 2t & 0 \\ 1 & 2t & -3 \end{vmatrix} = -3\sin 2t\boldsymbol{i}+3\cos 2t\boldsymbol{j}+(2t\cos 2t-\sin 2t)\boldsymbol{k}$$
  Putting it all together
  $$\frac{d(\boldsymbol{B}\times\boldsymbol{C})}{dt} = (-6t\cos 2t-3\sin 2t, -6t\sin 2t+3\cos 2t, -(2t^2+1)\sin 2t)$$

Question 2

$$\boldsymbol{r} = (a\cos(\omega t), b\sin(\omega t))$$

so we have velocity
$$\frac{d\boldsymbol{r}}{dt} = (-a\omega\sin(\omega t), b\omega\cos(\omega t))$$

and acceleration

$$\frac{d^2\boldsymbol{r}}{dt^2} = (-a\omega^2\cos(\omega t), -b\omega^2\sin(\omega t)) = -\omega^2\boldsymbol{r}$$

Let $|\boldsymbol{r}| = r$, and $\hat{\boldsymbol{r}}$ be a unit vector in the direction of $\boldsymbol{r}$. Then

$$\frac{d^2\boldsymbol{r}}{dt^2} = -\omega^2 r\hat{\boldsymbol{r}}$$

so the acceleration is in the opposite direction to $\boldsymbol{r}$, with magnitude proportional to r.

The speed is

$$\left|\frac{d\boldsymbol{r}}{dt}\right| = \omega\sqrt{a^2\sin^2(\omega t) + b^2\cos^2(\omega t)} = \omega\sqrt{b^2 + (a^2 - b^2)\sin^2(\omega t)}$$

since $\cos^2(\omega t) = 1 - \sin^2(\omega t)$.

Question 3

By Newton's second law

$$\frac{d^2\boldsymbol{r}}{dt^2} = -\frac{\mu\boldsymbol{r}}{r^3}$$

In a circular orbit, with radius a, we have $r = a$ and

$$\boldsymbol{r} = (a\cos(\omega t), a\sin(\omega t))$$

and by the previous question

$$\frac{d^2\boldsymbol{r}}{dt^2} = -\omega^2\boldsymbol{r}$$

Hence

$$\frac{-\mu\boldsymbol{r}}{a^3} = -\omega^2\boldsymbol{r}$$

that is to say

$$\omega^2 = \frac{\mu}{a^3}$$

Now, the period of a planet is

$$T = \frac{2\pi}{\omega} = \frac{2\pi a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}$$

Equivalantly, to fine μ:

$$\mu = \frac{\omega^2}{a^3} = \frac{4\pi^2 a^3}{T^2}$$

For the Earth

$$a = 1.5 \times 10^8 \text{ km} \qquad T = 365.25 \text{ days}$$

so

$$\mu = 9.987 \times 10^{20}\text{km}^3/\text{days}^2$$

Now, for Mercury, $a = 5.8 \times 10^7$ km, so the orbit is

$$T = \frac{2\pi a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} = 87.82 \text{ days}$$

Question 4

(i)

$$\frac{\partial f}{\partial x} = 2x^2 + 6xy + y^2 \qquad \frac{\partial f}{\partial y} = 3x^2 + 2xy + 12y^2$$

so

$$\frac{\partial^2 f}{\partial y \partial x} = 6x + 2y \qquad \frac{\partial^2 f}{\partial x \partial y} = 2x + 2y$$

(ii)

$$\frac{\partial f}{\partial x} = y^2 \ln(x^2 + y^2) + \frac{2x^2y^2}{x^2 + y^2} \qquad \frac{\partial f}{\partial y} = 2xy \ln(x^2 + y^2) + \frac{2xy^3}{x^2 + y^2}$$

so

$$\frac{\partial^2 f}{\partial y \partial x} = 2y \ln(x^2 + y^2) + \frac{2y^3}{x^2 + y^2} + \frac{4x^2y}{x^2 + y^2} - \frac{4x^2y^3}{(x^2 + y^2)^2}$$

and

$$\frac{\partial^2 f}{\partial x \partial y} = 2y \ln(x^2 + y^2) + \frac{4x^2y}{x^2 + y^2} + \frac{2y^3}{x^2 + y^2} - \frac{4x^2y^3}{(x^2 + y^2)^2}$$

(iii)

$$\frac{\partial f}{\partial x} = xy\cos(xy) + \sin(xy) \qquad \frac{\partial f}{\partial y} = x^2 \cos(xy)$$

so

$$\frac{\partial^2 f}{\partial y \partial x} = x\cos(xy) - x^2 y \sin(xy) + x\cos(xy) \qquad \frac{\partial^2 f}{\partial x \partial y} = 2x\cos(xy) - x^2 y \sin(xy)$$

(iv) First note that $r = \sqrt{x^2 + y^2}$ so

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r}$$

Hence

$$\frac{\partial f}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x} = \frac{x(r\cos r - \sin r)}{r^3}$$

and

$$\frac{\partial f}{\partial y} = \frac{y(r\cos r - \sin r)}{r^3}$$

Let

$$g(r) = \frac{r\cos r - r^2 \sin r}{r^3}$$

Then

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(xg) = x\frac{\partial g}{\partial y} = \frac{xy}{r}\frac{dg}{dr}$$

and

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(yg) = y\frac{\partial g}{\partial x} = \frac{yx}{r}\frac{dg}{dr}$$

(here we cleverly avoided having to work out everything explicitly).

Question 5

We have

$$\frac{\partial f}{\partial x} = 3\cos(x^2 - y^2) \times -\sin(x^2 - y^2) \times 2x = -6x\cos^2(x^2 - y^2)\sin(x^2 - y^2)$$

and

$$\frac{\partial f}{\partial y} = 6y\cos^2(x^2 - y^2)\sin(x^2 - y^2)$$

Hence

$$y\frac{\partial f}{\partial x} = -6xy\cos^2(x^2 - y^2)\sin(x^2 - y^2) = -x\frac{\partial f}{\partial y}$$

and

$$y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = 0$$

Question 6

We have $x = \ln r$, $r = \sqrt{x^2 + y^2}$, so

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r}$$

and

$$\frac{\partial z}{\partial x} = \frac{dz}{dr}\frac{\partial r}{\partial x} = \frac{x}{r^2} \qquad \frac{\partial z}{\partial y} = \frac{y}{r^2}$$

Now

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{r^2} - \frac{2x^2}{r^4} \qquad \frac{\partial^2 z}{\partial y^2} = \frac{1}{r^2} - \frac{2y^2}{r^4}$$

Hence

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2}{r^2} - \frac{2(x^2 + y^2)}{r^4} = \frac{2}{r^2} = \frac{2r^2}{r^4} = 0$$

since $x^2 + y^2 = r^2$.

Question 7

We have

$$\frac{\partial V}{\partial t} = -\frac{3x}{2t^{5/2}} \exp\left(\frac{x^2}{t}\right) - \frac{x^3}{t^{7/2}} \exp\left(\frac{x^2}{t}\right) = -\left(\frac{3}{2} + \frac{x^2}{t}\right) \frac{x}{t^{5/2}} \exp\left(\frac{x^2}{t}\right)$$

Now

$$\frac{\partial V}{\partial x} = \frac{1}{t^{3/2}} \exp\left(\frac{x^2}{t}\right) + \frac{2x^2}{t^{5/2}} \exp\left(\frac{x^2}{t}\right)$$

so

$$\begin{aligned}\frac{\partial^2 V}{\partial x^2} &= \frac{2x}{t^{5/2}} \exp\left(\frac{x^2}{t}\right) + \frac{4x}{t^{5/2}} \exp\left(\frac{x^2}{t}\right) + \frac{4x^3}{t^{7/2}} \exp\left(\frac{x^2}{t}\right) \\ &= 4\left(\frac{3}{2} + \frac{x^2}{t}\right) \frac{x}{t^{5/2}} \exp\left(\frac{x^2}{t}\right)\end{aligned}$$

We see that

$$\frac{\partial V}{\partial t} \Big/ \frac{\partial^2 V}{\partial x^2} = -\frac{1}{4}$$

which is constant