MAS165: Solutions to Chapter 1 Problems

Question 1

(i) $|\boldsymbol{c}| = \sqrt{(-1)^2 + 2^2 + 0^2} = \sqrt{5}$

(ii) $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = (4, -4, 0)$, so

$$|\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}| = \sqrt{4^2 + (-4)^2 + 0^2} = \sqrt{32} = 4\sqrt{2}$$

(iii) $2\boldsymbol{a} = 3\boldsymbol{b} - 5\boldsymbol{c} = 2(3, -2, 0) - 3(2, -4, 0) - 5(-1, 2, 0) = (5, -2, 0)$, so

$$|2\boldsymbol{a} - 3\boldsymbol{b} - 5\boldsymbol{c}| = \sqrt{5^2 + (-2)^2 + 0^2} = \sqrt{29}$$

Question 2

(a) $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = (2, -1, 4)$

(b) $\boldsymbol{b} \cdot \boldsymbol{c} = (-1 \times 1) + (2 \times -2) + (1 \times 1) = -4.$

(c) $2(\boldsymbol{a} + 2\boldsymbol{b}) + 3(\boldsymbol{a} - 2\boldsymbol{c}) = 2(0, 3, 4) + 3(0, 3, 0) = (0, 15, 8).$

(d) $2\boldsymbol{b} \cdot (\boldsymbol{a} - \boldsymbol{c}) = (-2, 4, 2) \cdot (1, 1, 1) = -2 + 4 + 2 = 4.$

(e) $\boldsymbol{a} \cdot \boldsymbol{b} = -2 - 2 + 2 = -2$, so

$$(\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} = -2(1, -2, 1) = (-2, 4, -2)$$

Question 3

- (a) and (d) add scalars to vectors, which is impossible; they have no meaning.
- (b) is a sum of scalars on the left, and a scalar on the right; it has meaning.
- (c) involves only vectors; it has meaning.

Question 4

We use

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. Now

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= \sqrt{3} + \sqrt{3} = 2\sqrt{3} \\ |\boldsymbol{a}| &= \sqrt{3+1} = 2 \\ |\boldsymbol{b}| &= \sqrt{1+3} = 2 \end{aligned}$$

so $2\sqrt{3} = 4\cos\theta$, ie:

$$\cos\theta = \frac{\sqrt{3}}{2}$$

From this it follows that $\theta = \frac{\pi}{6}$ (30 degrees).

Question 5

- $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if they are scalar multiples of each-other, or equivalently, $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$.
- $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Question 6

(a) $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$

(b)

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -1 & 2 \\ -1 & 2 & 1 \end{vmatrix} = -5\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k} = (-5, -4, 5)$$

(c)

$$\boldsymbol{b} \times \boldsymbol{c} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 4\boldsymbol{i} + 2\boldsymbol{j} = (4, 2, 0)$$

Question 7

As in question 4, if θ is the angle, then

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$$

Now

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= 2 + 6 + 12 = 20 \\ |\boldsymbol{a}| &= \sqrt{1+4+9} = \sqrt{14} \\ |\boldsymbol{b}| &= \sqrt{4+9+16} = \sqrt{29} \end{aligned}$$

so

$$\cos\theta = \frac{20}{\sqrt{13}\sqrt{29}} = 0.99258$$

(to 5 decimal places) and

$$\theta = \cos^{-1}(0.99258) = 0.12187$$

(6.82 degrees).

Question 8

$$\boldsymbol{r} \times \boldsymbol{s} == \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{vmatrix} = 3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k} = (3, -1, -2)$$

$$\boldsymbol{s} \times \boldsymbol{t} == \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 0 & 3 \\ 0 & 1 & 3 \end{vmatrix} = -3\boldsymbol{i} - 6\boldsymbol{j} + 2\boldsymbol{k} = (-3, -6, 2)$$

So

$$\boldsymbol{r} \cdot (\boldsymbol{s} \times \boldsymbol{t}) = (1, 1, 1) \cdot (-3, -6, 2) = -3 - 6 + 2 = -7$$

and

$$(\boldsymbol{r} \times \boldsymbol{s}) \cdot \boldsymbol{t} = (3, -1, -2) \cdot (-3, -6, 2) = -9 + 6 - 4 = -7$$

Question 9

We use the identity

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$$

Set $\boldsymbol{a} = \boldsymbol{c}$, so that

$$\boldsymbol{c} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{c} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{c} \cdot \boldsymbol{b})\boldsymbol{c}$$

Hence

$$(\boldsymbol{c} \times (\boldsymbol{b} \times \boldsymbol{c})) \times \boldsymbol{c} = (\boldsymbol{c} \cdot \boldsymbol{c})\boldsymbol{b} \times \boldsymbol{c} - (\boldsymbol{c} \cdot \boldsymbol{b})\boldsymbol{c} \times \boldsymbol{c} = c^2 \boldsymbol{b} \times \boldsymbol{c}$$

where $c = |\boldsymbol{c}|$.

Question 10

Let $\boldsymbol{p} = (1, 2, 1)$ be the position vector of P. Then

$$\boldsymbol{r} - \boldsymbol{p} = (1, 1, 1) + \lambda(2, -1, 3) - (1, 2, 1) = (2\lambda, -1 - \lambda, 3\lambda)$$

and if the distance from P to a point on the line is D, then

$$D^2 = |\boldsymbol{r} - \boldsymbol{p}|^2 = 4\lambda^2 + 1 + \lambda^2 + 2\lambda + 9\lambda^2 = 14\lambda^2 + 2\lambda + 1$$

The minimum value of D^2 (and hence D) occurs when

$$\frac{d(D^2)}{d\lambda} = 28\lambda + 2 = 0$$

that is $\lambda = -\frac{1}{14}$. Here we have

$$D^2 = 14\left(-\frac{1}{14}\right)^2 - \frac{2}{14} + 1 = \frac{13}{14}$$

so the minimum distance is

$$D = \sqrt{\frac{13}{14}}.$$

Question 11

The plane has equation

$$\boldsymbol{r} \cdot (3, -2, 1) = -1$$

which gives us normal vector $(3, -2, 1)$, and unit normal vector

$$\boldsymbol{n} = \frac{1}{\sqrt{14}}(3, -2, 1)$$

Observe that the point $A = (0, 0, -1)$ satisfies the equation $3x - 2y + z + 1 = 0$. Observe

$$\overrightarrow{AP} = (1, -2, 4)$$

The point P has position vetor $\boldsymbol{p} = (1, -2, 3)$, so the distance we require is

$$|\overrightarrow{AP} \cdots \boldsymbol{n}| = \frac{1}{\sqrt{14}}(3 + 4 + 4) = \frac{11}{\sqrt{14}}.$$