MAS165

Chapter 1: Problems

1. If $\mathbf{a} = (3, -2, 0)$, $\mathbf{b} = (2, -4, 0)$ and $\mathbf{c} = (-1, 2, 0)$, find the magnitudes of (i) $\mathbf{c}$, (ii) $\mathbf{a}+\mathbf{b}+\mathbf{c}$ and (iii) $2\mathbf{a}-3\mathbf{b}-5\mathbf{c}$.

2. If $\mathbf{a} = (2, -1, 2)$, $\mathbf{b} = (-1, 2, 1)$ and $\mathbf{c} = (1, -2, 1)$, find

   a) $\mathbf{a}+\mathbf{b}+\mathbf{c}$
   b) $\mathbf{b}\cdot\mathbf{c}$
   c) $2(\mathbf{a}+2\mathbf{b})+3(\mathbf{a}-2\mathbf{c})$
   d) $2\mathbf{b}\cdot(\mathbf{a}-\mathbf{c})$
   e) $(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

3. Decide which of the following expressions have any meaning:
   a) $\mathbf{a}+(\mathbf{b}\cdot\mathbf{c})$,
   b) $(\mathbf{a}+2\mathbf{c}+\mathbf{d})\cdot(\mathbf{a}+\mathbf{b})+|\mathbf{a}|^2=5$,
   c) $\mathbf{a}+(\mathbf{b}\times\mathbf{c})$,
   d) $\mathbf{a}\times\mathbf{b}+\mathbf{a}\cdot\mathbf{d}+\mathbf{a}\times\mathbf{d}$.

4. Find the angle between $\mathbf{a} = (\sqrt{3}, 1)$ and $\mathbf{b} = (1, \sqrt{3})$.

5. State the conditions for two vectors $\mathbf{a}$ and $\mathbf{b}$ to be (a) parallel and (b) perpendicular.

6. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are given as in Q2, find
   a) $\mathbf{a}\times\mathbf{a}$
   b) $\mathbf{a}\times\mathbf{b}$
   c) $\mathbf{b}\times\mathbf{c}$

and give their magnitudes.

7. Find the angle between the vectors $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (2, 3, 4)$.

8. If $\mathbf{r} = (1, 1, 1)$, $\mathbf{s} = (2, 0, 3)$ and $\mathbf{t} = (0, 1, 3)$, find $\mathbf{r} \times \mathbf{s}$ and $\mathbf{s} \times \mathbf{t}$ and hence show that

$$\mathbf{r} \cdot (\mathbf{s} \times \mathbf{t}) = (\mathbf{r} \times \mathbf{s}) \cdot \mathbf{t}$$

9. Show that $(\mathbf{c} \times (\mathbf{b} \times \mathbf{c})) \times \mathbf{c} = c^2 \mathbf{b} \times \mathbf{c}$.

10. Find the minimum distance from the point P with coordinates $(1, 2, 1)$ to the line $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$, where $\mathbf{a} = (1, 1, 1)$ and $\mathbf{b} = (2, -1, 3)$.

11. Find the minimal distance from the point P with coordinates $(1, -2, 3)$ to the plane $3x - 2y + z + 1 = 0$.

Chapter 2 Problems

1. Let $f(t) = t^2 - 3$ a scalar function and $\mathbf{A}(t) = (t, e^t, -t^2)$, $\mathbf{B}(t) = (\cos 2t, \sin 2t, 0)$ and $\mathbf{C}(t) = (t, t^2, -3t)$ are vectors. Calculate $d(f\mathbf{A})/dt$, $d(\mathbf{B} \cdot \mathbf{C})/dt$ and $d(\mathbf{B} \times \mathbf{C})/dt$.

2. A particle moves so that its position vector at the time t is $\mathbf{r} = (a \cos(\omega t), b \sin(\omega t))$, where a, b and ω are constants and $a > b$. Show that its acceleration is directed towards the origin and is of magnitude proportional to the distance of the particle from the origin. Show that the speed is given by

$$\omega \left[b^2 + (a^2 - b^2)(\sin \omega t)^2 \right]^{1/2}.$$

3. A planet P orbiting the sun S is acted upon by a force $\mathbf{F} = -\mu \mathbf{r}/r^3$ per unit mass, where μ is a positive constant and $\mathbf{r}$ is the vector $\vec{SP}$. If the orbit of the planet is a circle of radius a, show that the period of the planet is $2\pi a^{3/2}/\mu^{1/2}$. The Earth and Mercury are orbiting the sun, in approximately circular orbits. The Earth is at a mean distance of approximately 1.5×10^8km from the sun and Mercury at approximately 5.8×10^7km. Given that the Earth's period is 365.25 days, find the period of Mercury.

4. For each of the following functions, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Show that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

(i) $f(x, y) = x^3 + 3x^2y + xy^2 + 4y^3$,
(ii) $f(x, y) = xy^2 \ln(x^2 + y^2)$,
(iii) $f(x, y) = x \sin(xy)$,
(iv)

$$\frac{\sin r}{r}, \quad \text{where} \quad r^2 = x^2 + y^2.$$

5. If $f(x, y) = cos^3(x^2 - y^2)$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and show that

$$y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = 0.$$

6. If $z = \ln \sqrt{x^2 + y^2}$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

[This equation is called Laplace's equation in two dimensions. It arises in many physical applications.]

7. If $V(x, t) = \frac{x}{t^{3/2}} \exp\left(\frac{x^2}{t}\right)$, show that the ratio of $\frac{\partial V}{\partial t}$ to $\frac{\partial^2 V}{\partial x^2}$ is a constant.

Chapter 3 Problems

1. Calculate $\mathbf{F} = \text{grad} f = \nabla f$ for the following scalar functions:

(i) $f(x, y, z) = xyz$
(ii) $f(x, y, z) = xy + yz + xz$
(iii) $f(x, y, z) = 3x^2 - 4z^2$
(iv) $f(x, y, z) = e^{-x} \sin y$

2. Let $\phi = x^3 + xz + yz + 3$. Find gradϕ and the directional derivative of ϕ at the point (1,2,3) in the direction $\vec{r} = (1, 2, 0)$.

3. Find the divergence and the curl of the vectors $\mathbf{F}$ given in Q1.

4. Find the tangent plane and normal at the indicated point for the following surfaces:

(i) $x^2 + y^2 + z^2 = 4$; (1,1,$\sqrt{2}$)
(ii) $z^2 = x^2 - y^2$; $(1, 1, 0)$

5. Find the angle between the surfaces at the given points of intersection. (The angle between surfaces is the smaller of the 2 angles between their tangent planes at the point.)

(i) $z = 3x^2 + 2y^2$, $6x - y^2 - z = 0$, at $(1, 1, 5)$
(ii) $z = \sqrt{x^2 + y^2}$, $x^2 + y^2 = 4$, at $(2, 2, \sqrt{8})$.

6. It is given that the electric field $\mathbf{E}$ and the electrostatic potential V are connected by $\mathbf{E} = -\nabla V$. Find $\mathbf{E}$ for the case $V = \ln(x^2 + y^2)$. Verify that $\nabla^2 V = 0$.

7. The magnetic field $\mathbf{H}$ and the current density $\mathbf{J}$ are connected by Ampere's law, which, in appropiate units, is given by $\nabla \times \mathbf{H} = \mathbf{J}$. It is given that for $|y| \leq a$, $\mathbf{H}$ is given as

$$\mathbf{H} = H_0 \left(\frac{y^3}{a^3} \right) \mathbf{k}.$$

and for $|y| > a$, $\mathbf{H}$ is a constant vector where H_0 and a are positive constants. At $y = \pm a$ $\mathbf{H}$ is continious. Find $\mathbf{H}$ for $y > a$, $y < -a$, $\mathbf{J}$ for $|y| < a$ and show that $\mathbf{J} = 0$ for $|y| > a$.

8. Find a unit vector at the point (0,1,1), in the direction in which $\phi = x^2 - 3xy + 2y^2$ has its maximum rate of change, and find the magnitude of this rate of change.

9. The electric field $\mathbf{E}$ and the charge density ρ (i.e. the charge per unit volume) are connected by Gauss's law, which, in appropriate units is $\nabla \cdot \mathbf{E} = \rho$. It is given that $\mathbf{E} = E(x)\mathbf{i}$ and $\rho = 0$ for $x < -a$ and for $x > a$ and $\rho = 5\rho_0 x^4$ for $-a \leq x \leq a$, where ρ_0 is a constant. Find $E(x)$ given that $E(0) = 0$ and $E(x)$ is continuous at $x = \pm a$.

10. You are given that $\phi = e^{-r}/r^2$, where r is the distance from the origin. Show that

$$\nabla \phi = -e^{-r}\left(\frac{1}{r^3} + \frac{2}{r^4}\right)\mathbf{r}.$$

11. In spherical polars (r, θ, ϕ),

$$V = \left(r - \frac{a^3}{r^2}\right)\sin\theta \sin\phi.$$

Find ∇V in spherical polars.

12. The magnetic field $\mathbf{H}$ and the current density $\mathbf{J}$ are connected by Ampere's law which, in appropriate units, is $\nabla \times \mathbf{H} = \mathbf{J}$. In cylindrical polars (r, θ, z) you are given that for $r \leq a$

$$\mathbf{H} = H_0 \frac{r^2}{a^2}\hat{\theta},$$

and for $r > a$

$$\mathbf{H} = H_0 \frac{a}{r}\hat{\theta},$$

Find $\mathbf{J}$ for $r < a$ and for $r > a$.

13. Find the derivative of $ze^x \cos y$ at $(1, 0, \pi/3)$ in the direction of the vector $\mathbf{i} + 2\mathbf{j}$.

Chapter 4 Problems

1. Evaluate the double integral

$$\int\int_{\mathcal{R}} (x + 2y)dxdy,$$

where $\mathcal{R}$ is the triangular region bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$.

2. Evaluate

$$\int_0^1 dy \int_{2y}^2 (2x + y)dx.$$

Re-evaluate the same integral by reversing the order of integration.

3. Evaluate

$$\int\int_{\mathcal{R}} (2x + y)dxdy,$$

where $\mathcal{R}$ is the region in the first quadrant bounded by the curves $y = 0$, $x = 1$ and $y^2 = x$.

4. Transform into plane polars (r, θ), where $x = r\cos\theta$, $y = r\sin\theta$ and evaluate

$$\int\int_{\mathcal{R}} (x^2 + y^2)^{1/2}dxdy,$$

where $\mathcal{R}$ is the circular sector in the first quadrant bounded by $x^2 + y^2 = a^2$.

5. The surface density of a circular disc of radius a is $\sigma_0 r^2/a^2$, where σ_0 is a positive constant and r is the distance from the centre of the disc. Show that the mass of the disc is $\pi\sigma_0 a^2/2$.

6. Show that the volume V of the region that lies inside $z = x^2 + y^2$ and below the plane $z = 16$ is $V = 128\pi$.

7. A lamina of *uniform* surface density σ_0 occupies the region $\mathcal{R}$ between the semi-circle $x^2+y^2=4a^2$, $y>0$ and the line $y=a$ for values $y \geq a$. Show that the mass of the lamina is

$$\frac{\sigma_0 a^2\left(4\pi - 3\sqrt{3}\right)}{3}.$$

[Hint: You may assume that

$$\int \sqrt{4a^2-x^2}dx = \frac{x}{2}\sqrt{4a^2-x^2} + 2a^2\sin^{-1}\left(\frac{x}{2a}\right) + C.]$$

Chapter 5 Problems

1. A sphere of radius a is charged with charge density $q_0 r^3/a^3$, where r is the distance from the centre of the sphere, located at the origin of co–ordinates, and q_0 is a constant. Find the total charge of the sphere.

2. Given that $\mathbf{F}=(2x-y, 2y-z, x)$, evaluate the line integral

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$$

along the following paths $\mathcal{C}$ that join the point $O(0,0,0)$ and $A(1,1,1)$:

1. $x=t$, $y=t^2$ and $z=2t^2-t$
2. the straight line OA. Deduce that $\mathbf{F}$ is rotational.

3. Evaluate

$$\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r},$$

where $\mathbf{F}=(-ay, bx, 0)$ and $\mathcal{C}$ is the contour in the Oxy plane shown in the diagram and described in the direction shown by the arrow.

4. In cylindrical polars (r, θ, z), $\mathbf{H} = H_0 r^2 \hat{\theta}/a^2$, $r \leq a$, where H_0 and a are positive constants. Evaluate

$$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{r},$$

where $\mathcal{C}$ is the circle $z = 0$, $r = R$, described in the anticlockwise sense for $R < a$.

5. Evaluate the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{F} = (xz, 0, -yz)$ and $\mathbf{C}$ is the line segment from (-1,2,0) and (3,0,1).
[Hint: Find a suitable parametrization for the line segment.]

6. Evaluate

$$\int_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}} dS,$$

where $\mathbf{F} = (y + z, z + x, x + y))$ and $\mathcal{S}$ is the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, $z = 0$ and with $\hat{\mathbf{n}}$ pointing in the direction Oz.

7. In spherical polar coordinates $(r, \theta\phi)$ a vector field is given as $\mathbf{E} = E_0 (r/a)^2 \hat{\mathbf{r}}$ for $r < a$. Evaluate

$$\oint_{\mathcal{S}} \mathbf{E} \cdot \hat{\mathbf{n}} dS,$$

where $\mathcal{S}$ is the spherical surface with $r = a_1$ with $a_1 < a$.

8. Use *Gauss's divergence theorem* to evaluate

$$\oint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}} dS,$$

with $\mathbf{F} = (xy^2, y^3, y^2 z)$, where $\mathcal{S}$ is the closed surface of the cylinder $x^2 + y^2 = 4$, $0 \leq z \leq 2$.

9. Use *Gauss's divergence theorem* to evaluate

$$\oint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}} dS,$$

with $\mathbf{F} = (2xz, yz, z^2)$, where $\mathcal{S}$ is the closed surface of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

10. An electrostatic field $\mathbf{E}$ is given in Cartesian coordinates by

$$\begin{aligned} \mathbf{E} &= (E_0 x^3/a^3, 0, 0); && |x| \leq a \\ \mathbf{E} &= (-E_0, 0, 0); && x < -a \\ \mathbf{E} &= (E_0, 0, 0); && x > a, \end{aligned}$$

where E_0 is a constant. Find the charge density $\rho = \nabla \cdot \mathbf{E}$ for $x < a$ and for $|x| > a$. Verify Gauss's divergence theorem

$$\oint \mathbf{E} \cdot \hat{\mathbf{n}} dS = \int_\tau \nabla \cdot \mathbf{E} \, dV,$$

choosing the finite region τ that is bounded by the planes $x = \pm a_1$, $y = \pm b$, $z = \pm c$ for $a_1 < a$.

11. In cylindrical polars (r, θ, z), $\mathbf{H} = H_0 (r/a)^2 \hat{\theta}$ for $r \leq a$, where H_0 and a are positive constants. Evaluate $\mathbf{J} = \nabla \times \mathbf{H}$ for $r < a$. Verify Stokes' theorem

$$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{r} = \int_{\mathcal{S}} \mathbf{J} \cdot \hat{\mathbf{n}} \, dS,$$

where $\mathcal{C}$ is the circle $(z = 0,\ r = R)$, described in the anti-clockwise sense for $R < a$, and $\mathcal{S}$ is the plane surface bounded by $\mathcal{C}$.