A Rapid Introduction to Differential Topology

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Contents

1 Higher Dimensional Calculus

1.1 The Total Derivative

In order to talk about smooth manifolds, we need to be able to talk about smooth maps and differentation for functions from an open subset of $\mathbb{R}^m$ to $\mathbb{R}^n$. First, let us define the *norm* of a vector in $\mathbb{R}^n$ by the formula

$$\|(x^1, \dots, x^n)\| = \sqrt{(x^1)^2 + \cdots + (x^n)^2}.$$

The Euclidean metric on $\mathbb{R}^n$ is defined by the formula $d(x,y) = \|x - y\|$. This metric defines the topology used in the previous section when talking about open sets and homeomorphisms.

Recall that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *differentiable* at a point $x \in \mathbb{R}$ if the *derivative*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists.

We can rewrite this equation

$$f(x+h) = f(x) + f'(x)h + |h| r(h) \qquad \lim_{r \to 0} r(h) = 0$$

Definition 1.1 Let $U \subseteq \mathbb{R}^m$ be an open subset. We say that a function $\varphi\colon U \to \mathbb{R}^n$ is *differentiable* at a point $x \in U$ if there is a bounded linear map $(D\varphi)_x\colon \mathbb{R}^m \to \mathbb{R}^n$, and a function $r\colon V \to \mathbb{R}^n$ for some open set $V \subseteq \mathbb{R}^m$ containing 0 such that

$$\varphi(x+v) = \varphi(x) + (D\varphi)_x v + \|v\| r(v)$$

for all $v \in V$, where $\lim_{v \to 0} r(v) = 0$.

The above linear map $(D\varphi)_x\colon V \to W$ is called the *total differential* of the function φ at the point x.

We refer to a function $\varphi\colon U \to \mathbb{R}^n$ as *differentiable* if it is differentiable at every point of the domain U. The following is an exercise.

Proposition 1.2 *Any differentiable function $\varphi: U \to \mathbb{R}^n$ is continuous.*

It is straightforward to prove that the total differential at a point, if it exists, is unique. Further, if we have two functions $\varphi_1, \varphi_2: U \to \mathbb{R}^n$ that are both differentiable at a point $x \in U$, then for any $\alpha_1, \alpha_2 \in \mathbb{R}$, the function $y \mapsto \alpha_1\varphi_1(y) + \alpha_2\varphi_2(y)$ is also differentialble at x, with

$$D(\alpha_1\varphi_1 + \alpha_2\varphi_2)_x = \alpha_1(D\varphi_1)_x + \alpha_2(D\varphi_2)_x$$

Example 1.3 Let $T: V \to W$ be a bounded linear map. Then, for any point $x \in V$ and $v \in V$, we have

$$T(x+v) = T(x) + Tv = T(x) + Tv + \|v\| \cdot 0$$

It follows that the map T is differentiable, with total derivative T at any point $x \in V$.

The following is similarly easy to check.

Example 1.4 Let $c: R \to W$ be a constant map, where $R \subseteq V$ is an open set, and V and W are normed vector spaces. Then the map c is differentiable, and the total derivative is 0.

The following result is known as the *chain rule*. We omit the proof.

Theorem 1.5 *Let U, V, and W be normed vector spaces. Let $R \subseteq U$ and $S \subseteq V$ be open sets, and let $\varphi: R \to S$ and $\psi: S \to W$ be functions.*

Suppose that the function φ is differentiable at the point $x \in R$, and the function ψ is differentiable at the point $\varphi(x) \in S$. Then the composite $\psi \circ \varphi: R \to W$ is differentiable at the point $x \in R$, with total derivative

$$D(\psi \circ \varphi)_x = (D\psi)_{\psi(x)} \circ (D\varphi)_x$$

□

Definition 1.6 Let $U \subseteq \mathbb{R}^m$ be open. We call a map $\varphi: U \to \mathbb{R}^n$ a *smooth* if it is *infinitely differentiable*, ie: the function $D\varphi: \mathbb{R}^m \to \mathbb{R}^n$ is differentiable, the function $D(D\varphi): \mathbb{R}^m \to \mathbb{R}^n$ is differentiable, the function $D(D(D\varphi)): \mathbb{R}^m \to \mathbb{R}^n$ is differentiable, and so on.

More generally, let $X \subseteq \mathbb{R}^m$. We call a map $f: X \to \mathbb{R}^n$ smooth if there is an open neighbourhood $U \supseteq X$ and a smooth map $F: U \to \mathbb{R}^n$ such that $F|_U = f$.

Given subsets $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$, we call a smooth map $\varphi: X \to Y$ a *diffeomorphism* if there is a smooth inverse $\varphi^{-1}: Y \to X$.

Let $U, V \subseteq \mathbb{R}^n$ be open. Note that if $\varphi: U \to V$ is a diffeomorphism, for each $x \in \mathbb{R}^n$, by the chain rule we have that $D(\varphi \circ \varphi^{-1})_x = D(id)_x = I$, where $I: \mathbb{R}^n \to \mathbb{R}^n$ is the identity. Thus, for any diffeomorphism, the total derivative $(D\varphi)_x$ is invertible at each point $x \in U$.

The following result is called the *local diffeomorphism theorem* or *inverse function theorem*, and provides a partial converse.

Theorem 1.7 *Let $\varphi: U \to V$ be a smooth map between open subsets of $\mathbb{R}^n$. Let $x \in U$, and suppose $(D\varphi)_x$ is invertible. Then there is an open set $U' \ni x$ such that $\varphi|_{U'}: U' \to \varphi[U']$ is a diffeomorphism.* □

1.2 Differentiation of Paths

A *path* in $\mathbb{R}^n$ is simply a continuous function $\gamma\colon [a,b] \to \mathbb{R}^n$.

Definition 1.8 We say that the path γ is *differentiable* at the point $t \in [a,b]$ if the derivative

$$\gamma'(t) = \lim_{h\to 0} \frac{\gamma(t+h)-\gamma(t)}{h}$$

exists.

We call a path γ *smooth* if it is infinitely differentiable, that is to say *gamma*$'$ is differentiable, $(\gamma')'$ is differentiable, and so on.

We can rewrite the above equation

$$\gamma(t+h) = \gamma(t) + \gamma'(t)h + |h|r(h) \qquad \lim_{h\to 0} r(h) = 0$$

Thus, when $t \in (a,b)$, the above is related to the total derivative by the formula

$$(D\gamma)_t(h) = \gamma'(t)h$$

Let $V = \mathbb{R}^n$. Write $\gamma(t) = (\gamma^1(t),\ldots,\gamma^n(t)$. Then it is straightforward to check that γ is differentiable at t if and only if each γ_i is differentiable at t, and

$$\gamma'(t) = ((\gamma^1)'(t),\ldots,(\gamma^n)'(t))$$

It follows that a path γ is constant if and only if $\gamma'(t) = 0$ for all $t \in [a,b]$. This, along with the link with the total derivative, is the key to proving the following.

Theorem 1.9 *Let $U \subseteq \mathbb{R}^m$ be a* connected *open subset. Then a function $\varphi\colon U \to \mathbb{R}^n$ is constant if and only if it is differentiable at each point $x \in U$, with $(D\varphi)_x = 0$.*

Example 1.10 Define a path $\gamma\colon [0,2\pi] \to \mathbb{R}^2$ by the formula

$$\gamma(t) = (\cos t, \sin t)$$

Then γ is differentiable, and $\gamma'(t) = (-\sin t, \cos t)$.

In the above example, note that $\gamma(0) = \gamma(2\pi) = (1,0)$. So $\gamma(2\pi) - \gamma(0) = 0$. However, $\gamma'(t) \neq 0$ for all t, so there is no value $c \in [0,2\pi]$ such that

$$\gamma(2\pi) - \gamma(0) = (2\pi - 0)\gamma'(c)$$

The nearest analogy of the mean value theorem in higher dimensions is the following result, which we call the *mean value inequality.*

Theorem 1.11 *Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a differentiable function. Then there exists $c \in [a,b]$ such that*

$$\|\gamma(b) - \gamma(a)\| \leq (b-a)\|\gamma'(c)\|$$

□

1.3 Partial Derivatives

Let $U \subseteq \mathbb{R}^m$ be an open subset, and let $f\colon R \to \mathbb{R}$ be a map. Let $(x^1, \ldots, x^m)$ be coordinates in $\mathbb{R}^m$. Then the *partial derivative*, $\frac{\partial f}{\partial x^j}$ is defined by differentiating the function f with respect to the variable x^j while treating the other variables as constants.

More generally, if we have a function $f\colon U \to \mathbb{R}^n$, we can write

$$f(x^1, \ldots, x^m) = (f^1(x^1, \ldots, x^m), \ldots, f^n(x^1, \ldots, x^m))$$

and form the partial derivatives $\frac{\partial f^i}{\partial x^j}$.

Under many circumstances, we can swap the order of taking partial derivatives. The following is sometimes known as *Schwarz's theorem.*

Theorem 1.12 *Let $U \subseteq \mathbb{R}^m$ be an open subset, and let $(x^1, \ldots, x^m)$ be coordinates in the space $\mathbb{R}^m$. let $i, j \in \{0, \ldots, m\}$, and let $f\colon R \to \mathbb{R}$ be a map such that the second partial derivatives*

$$\frac{\partial^2 f}{\partial x^i \partial x^j}(x) \qquad \frac{\partial^2 f}{\partial x^j \partial x^i}(x)$$

exist and are continuous for all $x \in R$.

Then

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$$

□

Partial derivatives are useful for computations, but depend upon a choice of basis, and are therefore less useful theoretically.

Example 1.13 Define a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ by the formula

$$f(x, y) = \frac{xy^3}{x^2 + y^6} \qquad (x, y) \neq (0, 0)$$

and $f(0, 0) = 0$.

Then the partial derivatives $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$ exist. However, the function f is not continuous, and hence not differentiable, at the point $(0, 0)$.

The following is simply a case of manipulating the definitions.

Proposition 1.14 *Let $U \subseteq \mathbb{R}^m$ be an open subset, and let $f\colon U \to \mathbb{R}^n$ be differentiable at the point $x \in R$. Let $\{e_1, \ldots, e_m\}$ and $\{e'_1, \ldots, e'_n\}$ be the standard bases of the spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.*

Then we have

$$(Df)_x(e_j) = \frac{\partial f^1}{\partial x^j}(x)e'_1 + \cdots + \frac{\partial f^n}{\partial x^j}(x)e'_n$$

□

Hence the total derivative $(Df)_x$ has matrix $\left(\frac{\partial f^i}{\partial x^j}(x)\right)$ with respect to the standard bases. This matrix is sometimes called the *Jacobian matrix* of f.

Let $y = f(x)$, and suppose we have a map $g: f[U] \to \mathbb{R}^p$ that is differentiable at the point y. Abusing notation somewhat, let us write $(f^1, \ldots, f^n)$ to denote coordinates in the space $\mathbb{R}^n$. Then the total derivative $(Dg)_y$ has matrix $\left(\frac{\partial g^i}{\partial f^j}(y)\right)$, and by the chain rule the matrix of the derivative $D(g \circ f)_x$ is obtained by matrix multiplication of the above two matrices. In other words, the chain rule can be written

$$\frac{\partial (g \circ f)^i}{\partial x^j} = \sum_{k=1}^{n} \frac{\partial g^i}{\partial f^k} \frac{\partial f^k}{\partial x^j}.$$

Theorem 1.15 *let $U \subseteq \mathbb{R}^m$ be an open subset, and let $\varphi: U \to \mathbb{R}^n$. Let $(x^1, \ldots, x^m)$ be coordinates in $\mathbb{R}^m$, write $\varphi(x^1, \ldots, x^m) = (\varphi^1(x^1, \ldots, x^m), \ldots \varphi_n(x^1, \ldots, x^m))$, and let $x \in U$.*

Suppose that the partial derivatives of the function φ all exist on U, and are continuous at the point x. Then the total derivative $(D\varphi)_x$ exists, and is given by the matrix $\left(\frac{\partial \varphi^i}{\partial x^j}\right)$ with respect to the standard bases of the vector spaces $\mathbb{R}^m$ and $\mathbb{R}^n$. □

In particular, note that φ is smooth if and only if all partial derivatives of φ of all orders exist.

Example 1.16 Define a map $f: \mathbb{R}^2 \to \mathbb{R}^2$ by the formula

$$f(x, y) = (e^{x+y}, x^2 + y^2)$$

Write $f_1 = e^{x+y}$ and $f_2 = x^2 + y^2$. Then

$$\frac{\partial f_1}{\partial x} = e^{x+y} \qquad \frac{\partial f_1}{\partial y} = e^{x+y}$$

and

$$\frac{\partial f_2}{\partial x} = 2x \qquad \frac{\partial f_2}{\partial y} = 2y$$

These partial derivatives are all continuous. Hence the total differential is the linear transformation defined by the matrix

$$(Df)_{(x,y)} = \begin{pmatrix} e^{x+y} & e^{x+y} \\ 2x & 2y \end{pmatrix}$$

The following is an immediate consequence of theorem 1.15 and the local diffeomorphism theorem.

Corollary 1.17 *Let $f: U \to V$ be a map between subsets of $\mathbb{R}^n$. Let $x \in U$, and suppose that the matrix of partial derivatives $\left(\frac{\partial f^i}{\partial x^j}\right)$ exists and is continuous, and has non-zero determinant at the point $x \in U$. Then there is an open set $U' \ni x$ such that $\varphi|_{U'}: U' \to \varphi[U']$ is a diffeomorphism.* □

2 Manifolds

2.1 Topological Manifolds

A *manifold* of dimension n is a nice topological space that is locally homeomorphic to $\mathbb{R}^n$. More precisely, we have the following.

Definition 2.1 An *n-dimensional manifold* is a subspace $M \subseteq \mathbb{R}^N$ such that for every point $x \in X$ there are open sets $V \ni x$ and $U \subseteq \mathbb{R}^n$ together with a homeomorphism $\phi: U \to V$.

In section **??**, we shall see that the condition that $M \subseteq \mathbb{R}^N$ is not really needed, and any compact metric space that is locally homeomorphic to $\mathbb{R}^n$ is homeomorphic to a subspace of $\mathbb{R}^N$ for some N, and so is a manifold.

Definition 2.2 Let M be a topological space. A triple, (U, V, ϕ), where $U \subseteq \mathbb{R}^n$, $V \subseteq M$ are open subsets, and $\phi: U \to V$ is a homeomorphism is called a *chart*. A collection of charts $\{(\phi_\lambda, U_\lambda, V_\lambda) \mid \lambda \in \Lambda\}$ such that $\bigcup_{\lambda \in \Lambda} V_\lambda = M$ is called an *atlas* for M.

By definition, a subset of $\mathbb{R}^N$ is a manifold if and only if it has an atlas.

Example 2.3 Euclidean space $\mathbb{R}^n$ is a manifold of dimension n. There is an atlas consisting of a single chart, $(\mathbb{R}^n, \mathbb{R}^n, id)$.

Example 2.4 The sphere

$$S^n = \{(x^0, \dots, x^n) \in \mathbb{R}^{n+1} \mid (x^0)^2 + \cdots + (x^n)^2 = 1\}$$

is a manifold of dimension n. Observe that for any point $x \in S^n$ the space $S^n \backslash \{x\}$ is homeomorphic to $\mathbb{R}^n$. Consequently there is an atlas containing two charts.

Proposition 2.5 *Let M be an m-dimensional manifold, and let M' be an n-dimensional manifold. Then the Cartesian product, $M \times M'$ is an $(m+n)$-dimensional manifold.*

Proof: Let $M \subseteq \mathbb{R}^N$ and $M' \subseteq \mathbb{R}^{N'}$. Then $M \times M' \subseteq \mathbb{R}^{N+N'}$. Let

$$\{(U_\lambda, V_\lambda, \phi_\lambda) \mid \lambda \in \Lambda\}$$

and

$$\{(U'_\mu, V'_\mu, \phi'_\mu) \mid \mu \in \Lambda'\}$$

be atlases for the manifolds M and M' respectively.

Each product $U_\lambda \times U'_\mu$ is an open subset of Euclidean space $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. Hence the product $M \times M'$ is an $(m+n)$-dimensional manifold, with atlas

$$\{(U_\lambda \times U'_\mu, V_\lambda \times V'_\mu, (\phi_\lambda, \phi'_\mu)) \mid (\lambda, \mu) \in \Lambda \times \Lambda'\}$$

□

Example 2.6 The *torus*, T^2 is defined to be the product $S^1 \times S^1$ of two circles.

2.2 Smooth Manifolds

Definition 2.7 We call a subspace $M \subseteq \mathbb{R}^N$ a *smooth manifold* of dimension n if for every point $x \in X$ there are open sets $V \ni x$ and $U \subseteq \mathbb{R}^n$ together with a diffeomorphism $\phi: U \to V$.

The triple, (U, V, ϕ), where $U \subseteq \mathbb{R}^n$, $V \subseteq M$ are open subsets, and $\phi: U \to V$ is a homeomorphism is called a *smooth chart.* A collection of charts $\{(\phi_\lambda, U_\lambda, V_\lambda) \mid \lambda \in \Lambda\}$ such that $\bigcup_{\lambda\in\Lambda} V_\lambda = M$ is called a *smooth atlas* for M.

By definition, a subset of $\mathbb{R}^N$ is a smooth manifold if and only if it has a smooth atlas.

All of the examples in the previous section are in fact smooth manifolds. We will only look at one such in detail for now.

Example 2.8 The sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

is a smooth manifold of dimension 2. To see this, let $V_z^+ = \{(x, y, z) \in S^2 \mid z > 0\}$, $V_z^- = \{(x, y, z) \in S^2 \mid z < 0\}$, and define $V_x^\pm$, $V_y^\pm$ similarly. Then $\{U_x^\pm, U_y^\pm, U_z^\pm\}$ is an open cover of S^2.

Let $U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$, and define a diffeomorphism $\varphi: U \to V_z^+$ by

$$\varphi(x, y) = (x, y, \sqrt{1 - x^2 - y^2}).$$

Diffeomorphisms $U \to V_z^-$, $U \to V_x^\pm$, and $U \to V_y^\pm$ are defined similarly. So we have a smooth atlas for S^2.

Smooth maps between smooth manifolds are important in differential geometry. We usually check the smoothness of a map in terms of the 'coordinates' defined by the smooth charts. Specifically, by definition we have the following obvious result.

Proposition 2.9 *Let M and M' be manifolds, with smooth atlases $\{(\phi_\lambda, U_\lambda, V_\lambda) \mid \lambda \in \Lambda\}$ and $\{(\tilde{\phi}_\mu, \tilde{U}_\mu, \tilde{V}_\mu) \mid \mu \in M\}$ respectively.*

A map $f: M \to M'$ is smooth if and only if each map between open subsets of Euclidean space $\tilde{\phi}_\mu^{-1} \circ f \circ \phi_\lambda: \phi_\lambda f^{-1}[\tilde{V}_\mu] \cap U\lambda \to \tilde{U}_\mu \cap \tilde{\phi}_\mu^{-1} f[V_\lambda]$ is smooth. □

2.3 Paracompactness

Paracompactness is a technical property of metric spaces that is useful in joining functions defined locally to form a globally defined function. This idea is useful when it comes to generalising constructions on $\mathbb{R}^n$ to constructions on manifolds through charts.

Definition 2.10 Let X be a topological space, with open covers $\mathcal{U}$ and $\mathcal{V}$. The open cover $\mathcal{U}$ is said to be a *refinement* of the open cover $\mathcal{V}$ if every open set $U \in \mathcal{U}$ is contained in a set belonging to the collection $\mathcal{V}$.

The open cover $\mathcal{U}$ is said to be *locally finite* if each point $x \in X$ has an open neighbourhood that is contained in only finitely many open sets of the cover $\mathcal{U}$.

The space X is said to be *paracompact* if it is Hausdorff, and every open cover has a locally finite refinement.

Clearly any compact topological space is paracompact. The proof of the following is rather technical, however; we will not go into details here.

Theorem 2.11 *Any metric space is paracompact.* □

Paracompactness lets us define nice collections of maps called *partitions of unity*.

Definition 2.12 Let X be a topological space. Then the support, *supp*(f), of a function $f: X \to \mathbb{R}$ is the closure of the set of points $x \in X$ such that $f(x) \neq 0$.

Definition 2.13 Let $\{U_\lambda \mid \lambda \in \Lambda\}$ be a locally finite open cover of a space X. Then a collection of functions

$$\{\rho_\lambda: X \to [0,1] \mid \lambda \in \Lambda\}$$

is said to be a *partition of unity* subordinate to the cover $\{U_\lambda \mid \lambda \in \Lambda\}$ if:

- *supp*$(\rho_\lambda) \subseteq U_\lambda$
- For every point $x \in X$:

$$\sum_{\lambda \in \Lambda} \rho_\lambda(x) = 1$$

Our second major technical result on paracompactness is the following.

Theorem 2.14 *Let $\mathcal{U}$ be a locally finite open cover of a paracompact space X. Then there is a partition of unity on M subordinate to $\mathcal{U}$.* □

If M is a smooth manifold, we refer to a partition of unity on M consisting of smooth functions as a *smooth partition of unity.* The above theorem can be refined for smooth manifolds. Specifically, if $\mathcal{U}$ is a locally finite open cover of a smooth manifold, then there is a partition of unity on M subordinate to $\mathcal{U}$.

We now come to the promised result saying that the condition that $M \subseteq \mathbb{R}^N$ is not needed in a manifold. Actually, we will only prove this in the compact case.

Theorem 2.15 *Let M be a compact space for where for every point $x \in X$ there are open sets $V \ni x$ and $U \subseteq \mathbb{R}^n$ together with a homeomorphism $\phi: U \to V$. Then M is homeomorphic to a subset of some Euclidean space $\mathbb{R}^N$.*

Proof: As a compact space, M is paracompact. By compactness and paracompactness we can find a finite atlas

$$\{(U_i, V_i, \phi_i) \mid i = 1, \ldots, k\}$$

and maps $\rho_i: M \to [0,1]$ such that *supp*$(\rho_i) \subseteq V_i$ and $\sum_i \rho_i(x) = 1$ for every point $x \in M$.

The product $U_1 \times \cdots \times U_k$ is a subset of some Euclidean space $\mathbb{R}^N$. We can thus define a map $\phi: M \to \mathbb{R}^N$ by the formula

$$\phi(x) = (\rho_1(x)\phi_1^{-1}(x), \ldots, \rho_k(x)\phi_k^{-1}(x))$$

Now the map ϕ is an injective continuous map from a compact space to a Hausdorff space, and therefore a homeomorphism onto its image. □

Example 2.16 Let us define an equivalence relation, $\sim$, on the sphere S^n by identifying opposite points. Then *real projective space*, $\mathbb{RP}^n$, is defined to be the quotient

$$\mathbb{RP}^n = S^n / \sim$$

The space $\mathbb{RP}^n$ is a manifold of dimension n. Since the space $\mathbb{RP}^n$ is compact, by the above we need only construct an atlas.

Recall from example 2.8 that the sphere S^n is an n-dimensional manifold. Any point $x \in S^n$ has an open neighbourhood that does not contain the opposite point, $-x$. Consequently we can form an atlas

$$\{(U_\lambda, V_\lambda, \phi_\lambda) \mid \lambda \in \Lambda\}$$

for the sphere S^n such that if $x \in V_\lambda$ then $-x \notin V_\lambda$.

It is now easy to see how to proceed; let $q\colon S^n \to \mathbb{RP}^n$ be the quotient map. Then we can define an atlas

$$\{(U_\lambda, q[V_\lambda], q \circ \phi_\lambda) \mid \lambda \in \Lambda\}$$

for the space $\mathbb{RP}^n$.

An alternative description of projective space $\mathbb{RP}^n$ is as the set of one-dimensional subspaces of the vector space $\mathbb{R}^{n+1}$. Such a subspace determines a pair of opposite points on the sphere S^n.

2.4 Boundaries

Definition 2.17 We define the *upper half-space*, $\mathbb{R}^n_+$, to be the space:

$$\{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$$

An *n-dimensional manifold with boundary* is a subspace $M \subseteq \mathbb{R}^N$ such that for every point $x \in X$ there are open sets $V \ni x$ and $U \subseteq \mathbb{R}^n_+$ together with a homeomorphism $\phi\colon U \to V$.

An *n-dimensional smooth manifold with boundary* is a subspace $M \subseteq \mathbb{R}^N$ such that for every point $x \in X$ there are open sets $V \ni x$ and $U \subseteq \mathbb{R}^n_+$ together with a diffeomorphism $\phi\colon U \to V$.

We will sometimes write just 'manifold' when we really mean 'manifold with boundary'. When indulging in such an abuse of terminology, a manifold in the sense defined in the previous section (ie: a manifold with no boundary) is termed a *closed manifold*.

A triple (U, V, ϕ) in the above definition is again called a chart, and a collection of coordinate charts covering the space M is again called an atlas.

Definition 2.18 Let M be an n-dimensional manifold with boundary. Then we define the *boundary*, ∂M, to be the set of all points $x \in M$ that contain no neighbourhood homeomorphic to an open subset of the space $\mathbb{R}^n$.

The complement of the boundary, $M^0 = M \backslash \partial M$, is called the *interior*.

Thus, if $x \in \partial M$, and we have a chart $\phi\colon U \to V$, where $x \in V$, then $\phi(x^1, \ldots, x^{n-1}, 0) = x$.

A manifold with boundary, M, is not a closed manifold unless $\partial M = \emptyset$. However, the interior M^0 *is* a closed manifold, with the same dimension as that of the bounded manifold M.

Example 2.19 The half space, $\mathbb{R}^n_+$, is an n-dimensional manifold with boundary. The boundary is the space

$$\{(x^1, \ldots, x^{n-1}, 0) \mid x^i \in \mathbb{R}\}$$

which is just a copy of the Euclidean space $\mathbb{R}^{n-1}$.

Example 2.20 The closed disk,

$$\overline{D}^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} (x^i)^2 \leq 1\}$$

is an n-dimensional manifold with boundary. The boundary is the $(n-1)$-sphere, S^{n-1}.

The above two examples suggest the following result. Thr proof is straightforward.

Proposition 2.21 *Let M be an n-dimensional manifold with boundary. Then the boundary, ∂M, is an $(n-1)$-dimensional closed manifold. If M is smooth, then so is ∂M.* □

Proposition 2.22 *Let $f\colon M \to N$ be a homeomorphism between manifolds with boundary. Then $f[\partial M] = \partial N$.*

Proof: Let the manifold M (and so N) have dimension n. Let $x \in \partial M$. Then the point x has no neighbourhood homeomorphic to an open subset of the space $\mathbb{R}^n$. Since the map f is a homeomorphism, the point $f(x) \in N$ has no neighbourhood homeomorphic to an open subset of the space $\mathbb{R}^n$, and so $f(x) \in \partial N$.

Repeating the above argument with the homeomorphism f^{-1} tells us that $f[\partial M] = \partial N$. □

The corresponding result holds for diffeomorphisms between smooth manifolds.

Our next result lets us "glue together" manifolds along boundaries, and build many more examples. It is geometrically quite clear.

Proposition 2.23 *Let M_+ and M_- be n-dimensional manifolds with boundary. Suppose we have a homeomorphism $f\colon \partial M_+ \to \partial M_-$. Then the union*

$$M_+ \cup_f M_- = \frac{M_+ \coprod M_-}{x \sim f(x)}$$

is an n-dimensional manifold.

Further, if M_+ and M_- are smooth manifolds, and the map f is a diffeomorphsim, then the manifold $M_+ \cup_f M_-$ is also smooth. □

Example 2.24 Let M be a manifold. If we remove a small open neighbourhood homeomorphic to the open disk, we obtain a manifold with boundary. The boundary is homeomorphic to the sphere, S^n.

The *connected sum*, $M \sharp N$, of two manifolds M and N is defined by removing small open disks from each and joining together along the common boundary according to the above proposition.

The join of k tori is sometimes termed the *k-holed torus*.

3 The Tangent Bundle

3.1 Vector Bundles

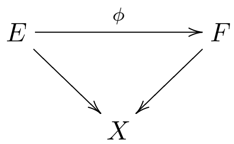
A vector bundle of dimension n over X is a space that locally resembles the product of X with an n-dimensional vector space. The precise definition is perhaps clearest when broken up into two parts, introducing some other relevant notions along the way.

Definition 3.1 A *pre-$\mathbb{R}^n$-bundle* over X is a space E equipped with a continuous *projection map* $p\colon E \to X$ such that for each point $x \in X$ the *fibre* $E_x = p^{-1}(x)$ is an n-dimensional real vector space, with the expected topology.

A pre-$\mathbb{R}^n$-bundle is also termed a *real pre-vector bundle of dimension n.*

Example 3.2 For any space X, $X \times \mathbb{R}^n$ is a $\mathbb{R}^n$-bundle, with projection map defined by the formula $p(x, v) = v$ where $x \in X$ and $v \in \mathbb{R}^n$. We call it the *product bundle* over X with dimension n.

Definition 3.3 Let E and F be pre-vector bundles over a space X. A *bundle map* is a continuous map $\phi\colon E \to F$ such that we have a commutative diagram



with the projection maps and the maps of fibres $\phi_x\colon E_x \to F_x$ are all linear. A *bundle isomorphism* is a bundle map whose inverse is also a bundle map.

We are now ready for our main definition.

Definition 3.4 An *$\mathbb{R}^n$-bundle* over a space X is a pre-$\mathbb{R}^n$-bundle such that for each point $x \in X$ there is an open neighbourhood U such that the restriction $E_U = p^{-1}[U]$ can be equipped with a bundle isomorphism $U \times \mathbb{R}^n \to E_U$.

An $\mathbb{R}^n$-bundle is also termed a *real vector bundle of dimension n.* It is similarly possible to define *$\mathbb{C}^n$-bundles*, or *complex vector bundles.*

A pair (U, ψ) where U is an open subset of X and $\psi\colon U \times \mathbb{R}^n \to E_U$ is a bundle isomorphism is called a *local trivialisation.* An *atlas of local trivialisations* for an $\mathbb{R}^n$-bundle E is a collection of local trivialisations $\{(U_\lambda, \psi_\lambda) \mid \lambda \in \Lambda\}$ such that $\bigcup_{\lambda \in \Lambda} U_\lambda = X$. By definition any $\mathbb{R}^n$-bundle has an atlas of local trivialisations. Such an atlas completely determines the structure of the bundle.

A one-dimensional vector bundle is sometimes referred to as a *line bundle.*

Definition 3.5 Let M be a smooth manifold. Then a vector bundle E over M is termed a *smooth vector bundle* if the space E is a smooth manifold, the projection map $p\colon E \to M$ is smooth, and E has an atlas of local trivialisations consisting of diffeomorphisms.

Example 3.6 Let $\mathbb{RP}^n$ be real projective space. We can view the space $\mathbb{RP}^n$ as the space of one-dimensional subspaces of the vector space $\mathbb{R}^n$. We define the *canonical line bundle*, E, over the space $\mathbb{RP}^n$ to be the subspace of the Cartesian product $\mathbb{RP}^n \times \mathbb{R}^{n+1}$ consisting of pairs (V, v) where $V \subseteq \mathbb{R}^{n+1}$ is a one-dimensional subspace, and $v \in V$. The structure map $p: E_n \to \mathbb{RP}^n$ is defined by the formula $p(V, v) = V$.

The following result provides a useful way of checking when a bundle map is an isomorphism. The proof depends on the fact that any vector bundle is locally trivial.

Proposition 3.7 *Let $\phi: E \to F$ be a bundle map. Then the map ψ is a bundle isomorphism if and only if each map of fibres $\psi: E_x \to F_x$ is an isomorphism of vector spaces.*

Proof: By definition each map of fibres $\phi_x: E_x \to F_x$ is a vector space isomorphism if the map ϕ is a bundle isomorphism.

To prove the converse, suppose initially that $E = X \times \mathbb{R}^n$ and $F = X \times \mathbb{R}^n$. Then we have a collection of vector space isomorphisms $\phi_x: \{x\} \times \mathbb{R}^n \to \{x\} \times \mathbb{R}^n$. The isomorphisms $T_x: \mathbb{R}^n \to \mathbb{R}^n$ defined by the formula

$$\psi(x, v) = (x, T_x(v))$$

depend continuously on the point $x \in X$. Hence the inverses, T_x^{-1}, also depend continuously on the point $x \in X$ and we have a bundle map $\phi^{-1}: E \to F$ defined by the formula

$$\phi^{-1}(x, v) = (x, T_x^{-1}(v))$$

In the general case, consider atlases of local trivialisations $\{(U_\lambda, \psi_\lambda^E) \mid \lambda \in \Lambda\}$ and $\{(U_\lambda, \psi_\lambda^F) \mid \lambda \in \Lambda\}$ for the bundles E and F respectively. Then for each element $\lambda \in \Lambda$ we have a bundle map

$$\tilde{\phi}_\lambda: U_\lambda \times \mathbb{R}^n \to U_\lambda \times \mathbb{R}^n$$

defined by forming the composition $(\psi_\lambda^F)\phi(\psi_\lambda^E)^{-1}$. By the above calculation, this bundle map has an inverse, $\tilde{\phi}_\lambda^{-1}$.

We can therefore define an inverse to the bundle map ϕ by writing

$$\phi^{-1}(v) = (\psi_\lambda^E)\phi_\lambda^{-1}(\psi_\lambda^F)^{-1}(v)$$

whenever $v \in F_{U_\lambda}$. □

The following result is proved similarly.

Proposition 3.8 *Let $\phi: E \to F$ be a smooth bundle map. Then the map ψ is a smooth bundle isomorphism if and only if each map of fibres $\psi: E_x \to F_x$ is an isomorphism of vector spaces.* □

Definition 3.9 Let E be an $\mathbb{R}^n$-bundle over a space X. Then the bundle E is called *trivial* if it is isomorphic to the product bundle $X \times \mathbb{R}^n$.

We can make a similar definition in the smooth case.

Example 3.10 Consider the canonical line bundle, E_1, over the projective line $\mathbb{RP}^1$ defined in example 3.6. Let $q: S^1 \to \mathbb{RP}^1$ be the quotient map identifying opposite points on the circle, S^1. Then the bundle E_1 is the set of points of the form

$$(q(\cos\theta, \sin\theta), t(\cos\theta, \sin\theta))$$

where $\theta \in [0, \pi]$ and $t \in \mathbb{R}$.

The quotient map q identifies the opposite ends of the strip $[0, \pi] \times \mathbb{R}$ under correspondence

$$(0, t) \mapsto (\pi, -t)$$

Thus the space E_1 is a Möbius band. In particular, the space E_1 is not homeomorphic to the product $\mathbb{RP}^1 \times \mathbb{R}$ (this is visually quite clear, though to prove it rigorously, we need to introduce the concept of orientation) so the bundle E_1 is not trivial.

3.2 Tangent Space

Definition 3.11 Let $M \subseteq \mathbb{R}^N$ be a smooth manifold of dimension n. Then we define the *tangent space*

$$T_xM = \{\gamma'(0) \in \mathbb{R}^N \| \gamma: (-\delta, \delta) \to M \text{ smooth }, \ \gamma(0) = x\}.$$

We define the *tangent bundle*

$$TM = \bigcup_{x \in M} \{x\} \times T_xM \subseteq M \times \mathbb{R}^N.$$

We have a smooth surjection $p: TM \to M$ given by the obvious formula $p(x, v) = x$, with fibres $p^{-1}(x) \cong T_xM$. We claim that TM is a real vector bundle of dimension n.

Proposition 3.12 *Pick a chart (U, V, ϕ) for M where $x \in V$. Set $\phi^{-1}(x) = (x^1, \ldots, x^n)$, and*

$$\gamma^i(t) = (x^1, \ldots, x^{i-1}, x^i + t, x^{i+1}, \ldots, x^n).$$

Set

$$\frac{\partial}{\partial x^i} = (\phi \circ \gamma^i)'(0).$$

Then

$$\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\}$$

is a basis for T_xM.

Proof: Let $v \in T_xM$. Then $v = \gamma'(0)$ for some smooth path $\gamma: (-\delta, \delta) \to V$ with $\gamma(0) = x$. Let $q = \phi^{-1}(x)$.

Define $y: (-\delta, \delta) \to U$ by $y = \phi^{-1} \circ \gamma$. Write

$$y(t) = (y^1(t), \ldots, y^n(t))$$

so $y^i(0) = x^i$. Set $\alpha^i = (y^i)'(0)$. Then

$$y'(0) = \alpha^1 e_1 + \cdots + \alpha^n e_n$$

where $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{R}^n$.

Now $\gamma = \phi \circ y$, so by the chain rule

$$v = \gamma'(0) = (D\phi)_q(y'(0)) = \alpha_1(D\phi)_q e_1 + \cdots + \alpha_n(D\phi)_1 e_n$$

Recall

$$\gamma^i(t) = (x^1, \ldots, x^{i-1}, x^i + t, x^{i+1}, \ldots, x^n).$$

so

$$\frac{\partial}{\partial x^i} = (\phi \circ \gamma^i)'(0) = (D\phi)_q(\gamma^i)'(0) = (D\phi)_q e_i.$$

Hence

$$v = \alpha^1 \frac{\partial}{\partial x^1} + \cdots + \alpha^n \frac{\partial}{\partial x^n}$$

and

$$\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\}$$

spans $T_x M$.

Now, suppose

$$\beta^1 \frac{\partial}{\partial x^1} + \cdots + \beta^n \frac{\partial}{\partial x^n} = 0$$

where $\beta^i \in \mathbb{R}$.

By the above

$$\frac{\partial}{\partial x^i} = (\phi \circ \gamma^i)'(0) = (D\phi)_q e_i.$$

so our equation tells us that

$$(D\phi)_q(\beta^1 e_1 + \cdots + \beta^n e_n) = 0$$

By the chain rule,

$$(D\phi^{-1})_x (D\phi)_q = \underset{\mathbb{R}^n}{id}$$

so, applying the linear map $(D\phi^{-1})_x$, we have

$$\beta^1 e_1 + \cdots + \beta^n e_n = 0.$$

The standard basis $\{e_1, \ldots, e_n\}$ is certainly linearly independent. Hence $\beta^i = 0$ for all i, and the set

$$\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\}$$

is also linearly independent. □

It follows that TM is a smooth pre-$\mathbb{R}^n$-bundle over M.

Proposition 3.13 *The manifold TM is a smooth $\mathbb{R}^n$-bundle over M.*

Proof: Let $x \in M$. Let (U, V, ϕ) be a chart, where $x \in V$. Write $\phi(x^1, \ldots, x^n) = x$. Then we can define a local trivalisation $\psi: V \times \mathbb{R}^n \to (TM)_V$ by

$$\phi(x, \alpha^1, \ldots, \alpha^n) = (x, \alpha_1 \frac{\partial}{\partial x^1} + \cdots + \alpha^n \frac{\partial}{\partial x^n}).$$

□

Let M be a smooth manifold, and $x \in M$. Suppose we have smooth charts (U, V, ϕ) and $(\tilde{U}, \tilde{V}, \tilde{\phi})$ such that $x = \phi(x_1, \ldots, x_n) = \tilde{\phi}(y_1, \ldots, y_n)$. Then we have two possible bases $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$ and $\left\{\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\right\}$ for $T_x M$. They are related by the following change of coordinate rule.

Proposition 3.14

$$\frac{\partial}{\partial x^j} = \sum_{i=1}^{n} \frac{\partial y_i}{\partial x^j} \frac{\partial}{\partial y_i}.$$

□

The proof is a straightforward application of the chain rule.

3.3 Differentiation

Before talking about differentials of smooth maps between manifolds, we need to extend the notion of a bundle map.

Definition 3.15 Let $f: X \to Y$ be a continuous map. Let E be a vector bundle over the space X, and let F be a vector bundle over the space Y. Then a *bundle map covering the map* f is a map $\psi: E \to F$ fitting into a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\psi} & F \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

such that each map of fibres $\psi_x: E_x \to F_{f(x)}$ is a homomorphism of vector spaces.

We can similarly define smooth bundle maps covering a smooth map.

Definition 3.16 Let $f: X \to Y$ be a continuous map. Let E be a vector bundle over the space Y with structure map p. Then we define the *pullback*, $f^\star(E)$, to be the subspace of the Cartesian product $E \times X$ consisting of pairs (v, x) such that $p(v) = f(x)$. The structure map is defined by the formula $p^\star(v, x) = x$.

It is straightforward to show that the above pullback is a vector bundle over the space X. It has fibres

$$f^\star(E)_x = E_{f(x)}$$

Proposition 3.17 *Suppose we have a bundle map $\psi: E \to F$ covering a map $f: X \to Y$ such that each map of fibres $\psi_x: E_x \to F_{f(x)}$ is an isomorphism. Then the vector bundle E is isomorphic to the pullback $f^\star[F]$.*

Proof: Let $v \in E_x$. Then $\phi(v) \in F_{f(x)} = f^\star(F)_x$. We can therefore define a bundle map $\psi' : E \to f^\star(F)$ by writing $\psi'(v) = \phi(v)$. The map ψ' is an isomorphism as it is an isomorphism on each fibre. □

The following is straightforward to prove.

Lemma 3.18 *Let M and N be manifolds, and let $f: M \to N$ be a smooth map. Let $x \in M$. Then we have a well-defined linear map $(Df)_x: T_xM \to T_xN$ given by the formula*

$$(Df)_x(\gamma'(0)) = (f \circ \gamma)'(0).$$

Let (U, V, ϕ) is a smooth chart for M where $x \in V$, and $x = \phi(x^1, \ldots, x^m)$, and let $(\tilde{U}, \tilde{V}, \tilde{\phi})$ be a smooth chart for N where $f(x) = \tilde{\phi}(y^1, \ldots, y^n)$. Then

$$\frac{\partial}{\partial x^j} = \sum_{i=1}^{m} \frac{\partial y_i}{\partial x^j} \frac{\partial}{\partial y_i}.$$

□

Proposition 3.19 *Let $f: M \to N$ be a smooth map. Then we have a bundle map $Df: TM \to TN$ covering f defined by the formula*

$$Df(x, v) = (f(x), (Df)_x(v)) \qquad x \in M,\ v \in T_xM.$$

□

The definition of smooth manifold along with the chain rule and the local diffeomorphism theorem for open subsets of $\mathbb{R}^n$ immediately gives us a generalisation of the local diffeomorphism theorem for manifolds.

Theorem 3.20 *Let M and N be smooth manifolds, and let $f: M \to N$ be a smooth map. Let $x \in M$, and suppose $(D\varphi)_x: T_xM \to T_{f(x)}N$ is invertible. Then there is an open set $U \ni x$ such that $f|_U: U \to f[U]$ is a diffeomorphism.* □

4 Critical and Regular Values

4.1 Submersions, Immersions, and Embeddings

Definition 4.1 Let $f: M \to N$ be a smooth map between manifolds. Then we call a point $x \in M$ a *critical point* if the differential $(Df)_x: T_xM \to T_{f(x)}N$ is not surjective.

Let M be an m-dimensional manifold, and N an n-dimensional manifold. Then $(Df)_x: T_xM \to T_{f(x)}N$ is a linear map from an m-dimensional vector space to an n-dimensional vector space. Thus, if $m < n$, every $x \in M$ is a critical point of f.

The notion is more meaningful if $m \geq n$.

Example 4.2 Let $f: M \to \mathbb{R}$ be a smooth map. Then a point $x \in M$ is a critical point if and only if the induced map $f: T_x M \to T_{f(x)}\mathbb{R}$ is zero. In terms of local coordinates $(x^1, \ldots, x^n)$ this means that:

$$\frac{\partial f}{\partial x^1} = 0 \ \cdots \ \frac{\partial f}{\partial x^n} = 0$$

Definition 4.3 A map $f: M \to N$ without critical points is called a submersion.

Example 4.4 The inclusion $i: U \hookrightarrow M$ of an open subset in a manifold is a submersion.

Example 4.5 Let $m > n$. The projection $p: \mathbb{R}^m \to \mathbb{R}^n$, defined by the formula

$$p(x^1, \ldots, x^m) = (x^1, \ldots, x^n)$$

is a submersion.

Proposition 4.6 *Let $f: M \to N$ be a smooth map. Suppose a point $x \in M$ is not a critical point. Then there is an open neighbourhood $U \ni x$ such that the restriction $f|_U: U \to N$ is a submersion.*

Proof: Since the point $x \in M$ is not critical, the linear transformation $(Df)_x: T_x M \to T_{f(x)} M$ has the greatest possible rank. The rank of a matrix is the size of the largest square submatrix with non-zero determinant.

Picking out charts in a neighbourhood of the space x and $f(x)$, and looking at the matrix of the transformation $(Df)_x$ with respect to the bases $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}\}$ and $\{\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\}$ for $T_x M$ and $T_{f(x)} M$ arising from these charts, the result follows once we note that the determinant function is continuous. □

Corollary 4.7 *The set of critical points of a smooth map is closed.* □

Definition 4.8 An *immersion* is a smooth map $f: M \to N$ such that for every point $x \in M$ the differential $(Df)_x: T_x M \to T_{f(x)} N$ is injective.

Example 4.9 The map $f: \mathbb{R} \to \mathbb{R}^2$ defined by the formula

$$f(x) = (x, |x|)$$

is not an immersion.

In general, immersions can exhibit some fairly strange behaviour. In order to cut down on such behaviour we make the following definition.

Definition 4.10 An immersion $f: M \to N$ is called an *embedding* if the map $f: M \to f[M]$ is a homeomorphism onto its image.

We call a subset $M \subseteq N$ an *embedded submanifold* if the inclusion $M \hookrightarrow N$ is an embedding. If $M \subseteq N$ is an embedded submanifold, we consider the tangent space $T_x M$ to be a subset of the tangent space $T_x N$.

Example 4.11 The inclusion $i: U \hookrightarrow M$ of an open subset in a manifold is an embedding.

Example 4.12 Let $m < n$. The map $i: \mathbb{R}^m \to \mathbb{R}^n$, defined by the formula

$$p(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0)$$

is an embedding.

Example 4.13 Let $M \subseteq \mathbb{R}^N$ be a smooth manifold. Then the inclusion $M \hookrightarrow \mathbb{R}^N$ is an embedding.

The following result is proved in the same way as proposition 4.6

Proposition 4.14 *Let $f: M \to N$ be a smooth map. Suppose we have a point $x \in M$ such that the differential $(Df)_x: T_xM \to T_{f(x)}M$ is injective. Then there is an open neighbourhood $U \ni x$ such that the restriction $f|_U: U \to N$ is a immersion.* □

Theorem 4.15 *Let $f: M \to N$ be a smooth map, and suppose that the differential $Df: T_xM \to T_{f(x)}N$ is injective. Then there are smooth charts (U, V, ϕ) and $(\tilde{U}, \tilde{V}, \tilde{\phi})$ where $x \in V$ and $f(x) \in \tilde{V}$ such that*

$$\tilde{\phi}^{-1} f \phi(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0)$$

Proof: Pick smooth charts (U_1, V_1, ϕ_1) and (U_2, V_2, ϕ_2) where $x \in V_1$ and $f(x) \in V_2$. Write

$$\phi_2^{-1} f \phi_1(x^1, \ldots, x^m) = (y^1, \ldots, y^n)$$

Since the differential $(Df)_x: T_xM \to T_{f(x)}N$ is injective, we can assume without loss of generality (by permuting the coordinates if necessary) that the determinant of the matrix

$$\det \left(\frac{\partial y_i}{\partial x^j} \right)_{i,j=1}^m$$

is non-zero at the point x.

Hence, by the local diffeomorphism theorem, the map $(x^1, \ldots, x^m) \mapsto (y^1, \ldots, y^m)$ is a diffeomorphism in some neighbourhood of the point $\phi_1^{-1}(x)$. We therefore have a smooth chart (U_3, V_3, ϕ_3) where $x \in V_3$, and

$$\phi_2^{-1} f \phi_3(y^1, \ldots, y^m) = (y^1, \ldots, y^m, z^{m+1}, \ldots, z^n)$$

Now, let us define a map g in a neighbourhood of the point $f(x)$ by the formula

$$\phi_3^{-1} g \phi_2(y^1, \ldots, y^n) = (y^1, \ldots, y^m, y^{m+1} - z^{m+1}, \ldots, y^n - z^n)$$

Then certainly

$$\phi_3^{-1} g \circ f \circ \phi_3(y^1, \ldots, y^m) = (y^1, \ldots, y^m, 0, \ldots, 0)$$

We need to check that the map g is a diffeomorphism in a neighbourhood of the point $f(x)$. Observe that the differential $D(\phi_3^{-1} g \phi_2)$ has determinant

$$\det \begin{pmatrix} 1 & 0 \\ \star & -1 \end{pmatrix} \neq 0$$

at the point $\phi_2^{-1} f(x)$. Therefore, by the local diffeomorphism theorem, the map g is indeed a diffeomorphism in a neighbourhood of the point $f(x)$, and we are done. □

The following result is proved similarly.

Theorem 4.16 *Let $f: M \to N$ be a smooth map, and suppose that a point $x \in M$ is not critical. Then there are smooth charts (U, V, ϕ) and $(\tilde{U}, \tilde{V}, \tilde{\phi})$ where $x \in V$ and $f(x) \in \tilde{V}$ such that*

$$\tilde{\phi}^{-1} f \phi(x^1, \ldots, x^m) = (x^1, \ldots, x^n)$$

□

Corollary 4.17 *Suppose that the manifolds M and N are closed. Let $f: M \to N$ be a smooth map, and let $y \in N$ be a regular value. Then the inverse image $f^{-1}(y)$ is an embedded submanifold of M of dimension $m - n$.*

Proof: If y is not in the image of f, the statement is vacuously true. Otherwise, choose a point $x \in f^{-1}(y)$. Then by the above, we have smooth charts (U, V, ϕ) and $(\tilde{U}, \tilde{V}, \tilde{\phi})$ such that $x \in V$, $y \in \tilde{V}$, and

$$\tilde{\phi}^{-1} f \phi(x^1, \ldots, x^m) = (x^1, \ldots, x^n)$$

Without loss of generality, suppose that $y = \tilde{\phi}(0, \ldots, 0)$. Then the set $V \cap f^{-1}(y)$ is the set of images under $\tilde{\phi}$ of points that take the form

$$(0, \ldots, 0, x^{n+1}, \ldots, x^m)$$

It follows that the inverse image $f^{-1}(y)$ is a manifold of dimension $m - n$. It is clear that the inclusion $f^{-1}(y) \hookrightarrow M$ is an embedding. □

A similar result applies to manifolds with boundary.

Corollary 4.18 *Let M and N be smooth manifolds with boundary. Let $f: M \to N$ be a smooth map, and let $y \in N$ be a regular value. Then the inverse image $f^{-1}(y)$ is an embedded submanifold of M, with dimension $m - n$ and boundary $f^{-1}(y) \cap \partial M$.* □

We now have a handy way to define manifolds by looking at smooth maps $f: \mathbb{R}^m \to \mathbb{R}^n$.

Example 4.19 Define $f: \mathbb{R}^{n+1} \to \mathbb{R}$ by the formula

$$f(x^1, \ldots, x^n) = \sum_i (x^i)^2$$

Then f is a smooth map, and 1 is a regular value. Therefore the sphere

$$S^{n-1} = f^{-1}(1)$$

is an n-dimensional embedded submanifold of $\mathbb{R}^n$.

4.2 Sard's Theorem

Let $f: M \to N$ be a smooth map. Sard's theorem is the statement that 'almost all' points of the manifold N are regular values. It is one of the most important tools in differential topology. In order to frame a more precise statement of the theorem, we need a small amount of measure theory.

Recall that we define the *volume* of a cuboid

$$Q = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$$

to be the product

$$Volume(Q) = \prod_{i=1}^{n} (b_i - a_i)$$

A subset $S \subseteq \mathbb{R}^n$ is said to have *measure zero* if for every $\varepsilon > 0$ we can find a sequence of cuboids $(Q_k)_{k=1}^{\infty}$ such that:

- $S \subseteq \bigcup_{k=1}^{\infty} Q_k$
- $\sum_k Volume(Q_k) < \varepsilon$

Example 4.20 Any countable set has measure zero. A countable union of sets of measure zero has measure zero.

Example 4.21 The hyperplane $\mathbb{R}^{m-1} \times \{0\}$ has measure zero as a subset of the space $\mathbb{R}^m$.

The following is straightforward; we will need it later on.

Proposition 4.22 *Consider a subset $K \subseteq \mathbb{R}^m$. Suppose that for every real number $t \in \mathbb{R}$ the intersection with the hyperplace $\mathbb{R}^{m-1} \times \{t\}$ has measure zero as a subset of the hyperplane $\mathbb{R}^{m-1} \times \{t\}$. Then the set K has measure zero as a subset of $\mathbb{R}^m$.* □

Definition 4.23 Let M be a smooth manifold of dimension n. Then a subset $S \subseteq M$ is said to have *measure zero* if there is a countable set of charts

$$\{(U_\lambda, V_\lambda, \phi_\lambda) \mid \lambda \in \Lambda\}$$

such that $S \subseteq \bigcup_{\lambda \in \Lambda} V_\lambda$, and each set $\phi_\lambda^{-1}(V_\lambda \cap S)$ has measure zero in $\mathbb{R}^n$.

The following topological fact sometimes makes it easier to prove that a set has measure zero.

Proposition 4.24 *Let M be a smooth manifold. Then M has a countable smooth atlas.* □

We are now ready to state Sard's theorem. Although we state it for closed manifolds, Sard's theorem is also valid for manifolds with boundary.

Theorem 4.25 *Let M and N be smooth closed manifolds. Let $f: M \to N$ be a smooth map. Then the set of critical values is a set of measure zero in the manifold N.*

In particular, a randomly chosen point in the manifold N will be a regular value of the map f. Of course, we consider a point of the manifold N that does not belong to the image of the map f to be a regular value.

By corollary 4.17, if the manifold M has dimension m, and the manifold N has dimension n, for a randomly chosen point $y \in N$, the inverse image $f^{-1}(y)$ is an embedded submanifold of dimension $m - n$.

Most of the hard work in the proof of Sard's theorem is contained in the following special case.

Theorem 4.26 *Let $U \subseteq \mathbb{R}^m$ be an open subset. Let $f: U \to \mathbb{R}^n$ be a smooth map. Then the set of critical values of f has measure zero.*

Proof of Sard's theorem: Let $f: M \to N$ be a smooth map. By proposition 4.24 we can find smooth countable atlases

$$\{(U_\lambda, V_\lambda, \phi_\lambda) \mid \lambda \in \Lambda\}$$

and

$$\{(\tilde{U}_\mu, \tilde{V}_\mu, \tilde{\phi}_\mu) \mid \mu \in M\}$$

for the manifolds M and N respectively.

The composites $\tilde{\phi}_\mu^{-1} \circ f \circ \phi_\lambda$ are all smooth, and a point $x \in M$ is a critical point if and only if the inverse image $\phi_\lambda^{-1}(x)$ is a critical point of the composite $\tilde{\phi}_\mu^{-1} \circ f \circ \phi_\lambda$ for some $\lambda \in \Lambda$ and $\mu \in M$. By theorem 4.26 the set of critical values of the composite $\phi_\lambda \circ f$ has measure zero.

The result now follows from the definition of a subset of a manifold being of measure zero. □

We now come to the proof of our special case. Let $U \subseteq \mathbb{R}^m$ be an open subset, and let $f: U \to \mathbb{R}^n$ be a smooth map. Let us write C to denote the set of critical points of f, and C_i to denote the set of points $x \in U$ such that all partial derivatives of order less than or equal to i vanish.

The proof of theorem 4.26 is then divided into three lemmas.

Lemma 4.27 *The image $f[C \backslash C_1]$ is a subset of measure zero in $\mathbb{R}^n$.*

Lemma 4.28 *The image $f[C_i \backslash C_{i+1}]$ is a subset of measure zero in $\mathbb{R}^n$.*

Lemma 4.29 *The image $f[C_m]$ is a subset of measure zero in $\mathbb{R}^n$.*

Proof of lemma 4.27: We work by induction on the dimension, m, of the domain. The lemma is clearly true when $m = 0$.

Consider a point $x \in C \backslash D$. Since the differential of the function f does not vanish at the point x, there is a projection $p: \mathbb{R}^n \to \mathbb{R}$ such that the composite $p \circ f$ has non-zero differential at x. Without loss of generality, let us assume that $p(x^1, \ldots, x^n) = x^n$.

The point x has an open neighbourhood, V, where the differential of the composite $p \circ f$ does not vanish. Thus every point $t \in \mathbb{R}$ is a regular value of the composite $p \circ f|_V$, and the inverse image

$$V_t = (p \circ f)^{-1}(t) \cap V$$

is a smooth submanifold of dimension $m - 1$.

By working in local coordinates, let us assume that $V_t \subseteq \mathbb{R}^{m-1}$. Suppose the conclusion of the lemma is true in dimension $m-1$. Then for each real number $t \in \mathbb{R}$ the set of critical values of the function $f|_{V_t}$ where the differential does not vanish is a set of measure zero in $\mathbb{R}^{m-1} \times \{t\}$.

Observe that a point $x \in V_t$ is a critical value of the function f if and only if it is a critical value of the restriction $f|_{V_t}$. Thus for every point $t \in \mathbb{R}$ the intersection

$$f[(C \backslash D) \cap V] \cap (\mathbb{R}^{m-1} \times \{t\})$$

is a set of measure zero in $\mathbb{R}^{m-1} \times \{t\}$. The desired result now follows from proposition 4.22 □

Proof of lemma 4.28: Again we work by induction on the dimension m. The result certainly holds when $m = 0$.

Choose a point $x \in U \backslash C_{i+1}$. Let g be an order i derivative of the function f such that the differential of g at x is non-zero. Then we can find an open neighbourhood of the point x in which the derivative of g is non-vanishing. Thus the set $U \backslash C_{i+1}$ is open.

Let us consider the restriction $g|_{U \backslash C_{i+1}}$. By construction, the point 0 is a regular value of the function $g|_{U \backslash C_{i+1}}$. Let $V = (g|_{U \backslash E_{i+1}})^{-1}(0)$. Then V is an embedded submanifold of the space U of dimension $m - 1$.

By construction, $g(x) = 0$ for all points $x \in C_i \backslash C_{i+1}$. Therefore the set $E_i \backslash E_{i+1}$ is a subset of a manifold of dimension $m-1$. Looking at local coordinates, we can assume that the set $C_i \backslash C_{i+1}$ is a subset of $\mathbb{R}^{m-1}$.

Thus if the desired result holds in dimension $m-1$, it also holds in dimension m. This completes the process of induction. □

Proof of lemma 4.29: Let $x \in U$. Then we can write

$$f(x) = (f_1(x), \dots, f_n(x))$$

Let E be the set of points $x \in U$ of the function f_1 such that all partial derivatives of order less than or equal to m vanish at the point x. Then:

$$f[C_m] \subseteq f_1[E] \times \mathbb{R}^{n-1}$$

It therefore suffices to show that the set $f_1[E]$ has measure zero in $\mathbb{R}$. We proceed by proving that the image $f[E_m \cap Q]$ has measure zero for any m-dimensional cube $Q \subseteq U$.

Let s be the length of a side of the cube Q, and choose a positive integer $k \in \mathbb{N}$. Then we can partition the cube Q into k^m smaller cubes, each with sides of length s/k. The diameter of one of these smaller cubes is $s\sqrt{m}/k$.

Consider a point $x_0 \in E_m \cap Q$. Let Q_0 be one of the smaller cubes containing the point x_0. Observe that the set of partial derivatives of the function f of order at most $m+1$ is bounded on the cube Q.[1] Hence by Taylor's theorem there is a constant $C > 0$ such that:

$$|f_1(x) - f_1(x_0)| \leq C\|x - x_0\|^{m+1}$$

[1]Because the partial derivatives are all continuous and the cube is compact.

whenever $x \in Q$. In particular, if $x \in Q_0$ then:

$$|f_1(x) - f_1(x_0)| \leq C \left(\frac{s\sqrt{m}}{k} \right)^{m+1}$$

Thus the image $f_1(Q_0)$ is contained in an interval of length A/k^{m+1} where A is some fixed constant. The cube Q is made up of k^m cubes of the same size as the cube Q_0. Therefore the image $f[Q \cap E_n]$ is contained in a union of intervals of length at most:

$$k^m \frac{A}{k^{m+1}} = \frac{A}{k}$$

By increasing the natural number k, the length A/k can be made arbitrarily small. It follows that the set $f_1[Q \cap E_1]$ has measure zero. □

4.3 The Brouwer Fixed Point Theorem

The following result is called *Hirsch's theorem* after it's discoverer.

Theorem 4.30 *Let M be a smooth manifold with boundary, ∂M. Then there is no smooth map $f \colon M \to \partial M$ such that $f(x) = x$ for all $x \in \partial M$.*

Proof: Suppose that $f \colon M \to \partial M$ is a smooth map where $f(x) = x$ for all $x \in \partial M$. We seek a contradiction.

By Sard's theorem, the set of critical values of f in ∂M has measure zero, so we have at least one regular value $y \in \partial M$. Then $f|_{\partial M}$ is the identity map, so y is certainly also a regular value of $f|_{\partial M}$. If M has dimension n, then ∂M has dimension $n-1$, so by corollary 4.18, $f^{-1}(y)$ is an embedded manifold of dimension 1, with boundary

$$f^{-1}(y) \cap \partial M = \{y\}.$$

But $f^{-1}(y)$ is also compact. The only compact one-dimensional manifolds with boundary are disjoint unions of closed line segments and circles. Thus the boundary of $f^{-1}(y)$ must have an even number of points. This is a contradiction. □

Now consider the disk

$$D^{n+1} = \{(x^0, x^1, \ldots, x^n) \in \mathbb{R}^{n+1} \mid (x^0)^2 + (x^1)^2 + \cdots + (x^n)^2 \leq 1$$

This is an $(n+1)$-dimensional manifold with boundary

$$S^n = \{(x^0, x^1, \ldots, x^n) \in \mathbb{R}^{n+1} \mid (x^0)^2 + (x^1)^2 + \cdots + (x^n)^2 = 1$$

Lemma 4.31 *Let $g \colon D^{n+1} \to D^{n+1}$ be a smooth map. Then there is a point $x \in D^{n+1}$ such that $g(x) = x$.*

Proof: Suppose $g(x) \neq x$ for all $x \in D^{n+1}$. Define a smooth map $f \colon D^{n+1} \to S^n = \partial D^{n+1}$ by setting $f(x)$ to be the point on S^n obtained by following a line from $g(x)$ through x.

Then $f(x) = x$ if $x \in S^n$, which does not exist by the above. We conclude that $g(x) = x$ for some $x \in D^{n+1}$. □

A point $x \in D^{n+1}$ such that $g(x) = x$ is called a *fixed point* of g.

The *Weierstrass approximation theorem* asserts that given a continuous function $f: [a, b] \to \mathbb{R}$, and a number $\varepsilon > 0$, there is a polynomial $p: [a, b] \to \mathbb{R}$ such that $|p(x) - f(x)| < \varepsilon$ for all $x \in [a, b]$.

Note in particular that polynomials are smooth. A careful argument using the Weierstrass aproximation theorem, combined with the above lemma, yields the following result.

Theorem 4.32 (The Brouwer approximation theorem) *Let $g: D^{n+1} \to D^{n+1}$ be a continuous map. Then there is a point $x \in D^{n+1}$ such that $g(x) = x$.* □

5 Vector and Tensor Fields

5.1 Sections

Definition 5.1 A *section* of a vector bundle E over a space X is a continuous map $u: M \to E$ such that $u(x) \in E_x$ for all points $x \in M$.

A set of sections $\{u_1, \ldots, u_n\}$ is said to be *linearly independent* if and only if the set of vectors $\{u_1(x), \ldots, u_n(x)\}$ is linearly independent for all points $x \in M$.

We write $\Gamma(E)$ for the set of sections on a vector bundle E. If E is a smooth vector bundle, we write $\Gamma^\infty(E)$ to denote the set of smooth sections.

Proposition 5.2 *Let E be an $\mathbb{R}^n$-bundle over a space X. Then the bundle E is trivial if and only if we can find a set $\{u_1, \ldots, u_n\}$ of linearly independent sections.*

Proof: Suppose that the bundle E is trivial. Then there is a bundle isomorphism $\phi: E \to X \times \mathbb{R}^n$. Let $e_1, \ldots, e_n$ be the standard basis for the space $\mathbb{R}^n$. Then we can define a set of linearly independent sections $\{u_1, \ldots, u_n\}$ by writing

$$u_i(x) = \psi^{-1}(x, e_i)$$

Conversely, suppose we have a set of linearly independent sections $\{u_1, \ldots, u_n\}$. Then we can define a bundle map $\phi: E \to M \otimes \mathbb{R}^n$ by the formula

$$\phi(\alpha_1 u_1(x) + \cdots + \alpha_n u_n(x)) = (x, \alpha_1 e_1 + \cdots + \alpha_n e_n)$$

The map ϕ is an isomorphism of vector bundles by proposition 3.7. □

In particular, a line bundle is trivial if and only if it has a nowhere-vanishing section. The above result has a smooth analogue.

Proposition 5.3 *Let E be a smooth $\mathbb{R}^n$-bundle over a manifold M. Then the bundle E is trivial if and only if we can find a set $\{u_1, \ldots, u_n\}$ of linearly independent smooth sections.* □

Example 5.4 Recall that we define projective space, $\mathbb{RP}^n$ by identifying opposite points on the sphere S^n, or alternatively as the set of one-dimensional subspaces of the vector space $\mathbb{R}^{n+1}$, since such a subspace determines a pair of opposite points on the sphere S^n. In example 3.6 we defined the canonical line bundle, E, over $\mathbb{RP}^n$ to be the space

$$\{(v, V) \mid V \in \mathbb{RP}^n,\ v \in V\}$$

where the fibre at $V \in \mathbb{RP}^n$ is the one-dimensional space V itself.

The canonical line bundle, E_n, over projective space $\mathbb{RP}^n$ defined in example 3.6 is non-trivial. To see this, we appeal to the above result.

Suppose we have a section $s: \mathbb{RP}^n \to E$. Let $q: S^n \to \mathbb{RP}^n$ be the map associating a point on the sphere with the one-dimensional subspace of $\mathbb{R}^{n+1}$ containing that point, and consider the composition $sq: S^n \to E$. Then a point $x \in S^n$ is mapped to the pair

$$(q(x), t(x)x) \in E$$

for some continuous function $t: \mathbb{RP}^n \to \mathbb{R}$ such that $t(-x) = t(x)$.

By the intermediate value theorem there must be a point $x \in S^1$ such that $t(x) = 0$. Thus there must be a point at which the section s is equal to zero, and therefore the bundle E is non-trivial.

We also need to consider vector bundles equipped with extra structure.

Definition 5.5 Let E be a vector bundle over a space X. Then a *metric* on E consists of an inner product on each fibre E_x such that for any two sections $u, v \in \Gamma(E)$ the map

$$x \mapsto \langle s(x), t(x) \rangle$$

is continuous.

If E is a smooth vector bundle over a manifold M, a metric $\langle -, - \rangle$ is said to be *smooth* if for any two smooth sections $u, v \in \Gamma^\infty(E)$ the map

$$x \mapsto \langle s(x), t(x) \rangle$$

is smooth.

Theorem 5.6 *Let E be a smooth vector bundle over a manifold M. Then the bundle E can be equipped with a smooth metric.*

Proof: To begin, observe that by theorem **??** we can find a smooth locally finite atlas of local trivialisations $\{(U_\lambda, \psi_\lambda) \mid \lambda \in \Lambda\}$. We can define a metric on the restriction E_{U_λ} by writing

$$\langle e_1, e_2 \rangle_\lambda = \langle v_1, v_2 \rangle$$

where $\psi_\lambda(e_i) = (x, v_i)$ for points $e_i \in E_x$ and $v_i \in \mathbb{R}^n$, and the inner product $\langle v_1, v_2 \rangle$ is the standard inner product of the vectors v_1 and v_2 in the space $\mathbb{R}^n$.

Choose a smooth partition of unity $\{\rho_\lambda \mid \lambda \in \Lambda\}$ subordinate to the the open cover $\{U_\lambda \mid \lambda \in \Lambda\}$. We can define a metric on the bundle E by writing:

$$\langle e_1, e_2 \rangle = \sum_{\lambda \in \Lambda} \rho_\lambda(x) \langle e_1, e_2 \rangle_\lambda$$

for all points $e_1, e_2 \in E_x$. □

5.2 Vector Fields

Definition 5.7 A *vector field* on a manifold M is a smooth section of the tangent bundle TM.

By proposition 5.2, if M is an n-dimensional manifold, the tangent bundle TM is trivial if and only if there are n linearly independent smooth vector fields. A manifold with a trivial tangent bundle is called *parallelisable*.

Example 5.8 The sphere, S^3, can be viewed as the set of unit quarternions:

$$S^3 = \{w \in \mathbb{H} \mid |w| = 1\}$$

We have a smooth surjection $f\colon \mathbb{R}^3 \to S^3$ defined by the formula

$$f(\theta, \phi, \psi) = (\cos\theta + i\sin\theta)(\cos\phi + j\sin\phi)(\cos\psi + k\sin\psi)$$

There are three linearly independent vector fields on the sphere S^3 defined by the formulae:

$$\begin{array}{rcl} X(f(\theta,\phi,\psi)) & = & f_\star(\frac{\partial}{\partial\theta}) \\ Y(f(\theta,\phi,\psi)) & = & f_\star(\frac{\partial}{\partial\phi}) \\ Z(f(\theta,\phi,\psi)) & = & f_\star(\frac{\partial}{\partial\psi}) \end{array}$$

Therefore the sphere S^3 is parallelisable.

A similar argument, left as an exercise, can be used to show that the sphere S^7 is parallelisable. However, it is a remarkable result due to Adams that S^1, S^3 and S^7 are the only spheres which are parallelisable.

Recall that the tangent space T_xM is defined to be the set of all derivatives $\gamma'(0)$ where $\gamma\colon(-\delta,\delta) \to M$ is a smooth function with $\gamma(0) = x$. The following therefore makes sense.

Definition 5.9 Let $X \in \Gamma^\infty(TM)$ be a vector field, and let $f \in C^\infty(M)$ be as smooth function. Then we define the *derivative* of f *along* X to be the smooth function defined by writing

$$X(f)(x) = (f \circ \gamma)'(0) \quad \gamma\colon(-\delta,\delta) \to M,\ \gamma(0) = x,\ \gamma'(0) = X(x).$$

Let (U, V, ϕ) be a chart. For $x \in M$, write $\phi(x^1, \ldots, x^n) = x$ and

$$X(x) = X^1 \frac{\partial}{\partial x^1} + \cdots + X^n \frac{\partial}{\partial x^n}$$

where $X^1, \ldots, X^n$ are smooth real-valued functions.

Then by the chain rule

$$X(f) = X^1 \frac{\partial f \circ \phi}{\partial x^1} + \cdots + X^n \frac{\partial f \circ \phi}{\partial x^n}$$

By definition, a vector field $X \in \Gamma^\infty(TM)$ is determined by the operator

$$f \mapsto X(f)$$

on smooth maps $f \in C^\infty(M)$.

Definition 5.10 We define the *Lie bracket*, $[X, Y]$, of vector fields X and Y by the formula

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

where $f \in C^\infty(M)$.

Proposition 5.11 *The Lie bracket, $[X, Y]$, is a vector field.*

Proof: With respect to some chart ϕ, let us write:

$$X = \sum_i X^i \frac{\partial}{\partial x^i} \qquad Y = \sum_j Y^j \frac{\partial}{\partial x^j}$$

Then:

$$X(Y(f)) = \sum_{i,j} X^i \frac{\partial}{\partial x^i} \left(Y^j \frac{\partial f \circ \phi}{\partial x^j} \right) = \sum_{i,j} X^i Y^j \frac{\partial^2 f \circ \phi}{\partial x^i \partial x^j} + X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial f \circ \phi}{\partial x^j}$$

Similarly:

$$Y(X(f)) = \sum_{i,j} X^i Y^j \frac{\partial^2 f \circ \phi}{\partial x^j \partial x^i} + Y^i \frac{\partial X^j}{\partial x^i} \frac{\partial f \circ \phi}{\partial x^j}$$

Thus we have the formula:

$$[X, Y] = \sum_{i,j} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$$

which defines a vector field. □

5.3 Integral Curves and Flows

A *flow* is a certain type of continuous deformation of a manifold.

Definition 5.12 Let M be a smooth manifold. A *flow* on M is a smooth map $\theta: \mathbb{R} \times M \to M$ such that:

- $\theta(0, x) = x$ for all $x \in M$
- $\theta(s + t, x) = \theta(s, \theta(t, x))$ for all $s, t \in \mathbb{R}$ and $x \in M$

The second of the above conditions can be written

$$\theta(s + t, -) = \theta(s, -) \circ \theta(t, -)$$

Note that the above definition tells us that each map $\theta(s, -): M \to M$ is a diffeomorphism, with inverse $\theta(-s, -)$. For this reason a flow on a manifold M is sometimes called a *one parameter group of diffeomorphisms*.

Example 5.13 For any manifold M we can define the *constant flow* $\theta: \mathbb{R} \times M \to M$ by writing

$$\theta(t, x) = x$$

Example 5.14 We can define a flow $\theta: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by the formula

$$\theta(s, x) = s + x$$

Most of the interesting examples of flows come from looking at vector fields.

Definition 5.15 The *tangent field* of a flow θ on a manifold M is the vector field X defined by writing

$$X(f)(x) = \lim_{h \to 0} \left[\frac{f(\theta(h, x)) - f(x)}{h} \right]$$

where $f \in C^\infty(M)$ and $x \in M$.

Example 5.16 The tangent field of a constant flow is the zero vector field.

Example 5.17 Consider the flow $\theta: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by the formula

$$\theta(s, x) = s + x$$

The tangent vector field is defined by writing

$$X(x) = \frac{d}{dx}$$

Let us say that a vector field is *compactly supported* if it is zero outside of some compact set.

Theorem 5.18 *Let X be a compactly supported vector field on a closed manifold M. Then there is a unique flow on M with tangent field X.*

Proof: Consider a smooth curve $c: \mathbb{R} \to M$. We can define the *velocity vector* $dc/dt \in T_{c(t)}M$ by the formula

$$\frac{dc}{dt} = \lim_{h \to 0} \left[\frac{f(c(t+h)) - f(c(t))}{h} \right]$$

Suppose that θ is a flow with tangent field X. Then for each point $x \in M$ we can write:

$$\frac{d\theta(t, x)}{dt}(f) = \lim_{h \to 0} \left[\frac{f(\theta(t+h, x)) - f(\theta(t, x))}{h} \right] = X(f)(\theta(t, x))$$

since $\theta(t+h, x) = \theta(h, \theta(t, x))$.

Thus we have the differential equation

$$\frac{d\theta(t, x)}{dt} = X(\theta(t, x)) \qquad \theta(0, x) = x$$

With respect to a chart $(U, V\phi)$, let us write $x = \phi(x^1, \ldots, x^n)$, $X = X^1 \frac{\partial}{\partial x^1} + \cdots + X^n \frac{\partial}{\partial x^n}$, and $\theta(t, x) = \phi(\theta^1, \ldots, \theta^n)$. Then we have a system of equations

$$\frac{d\theta^i(t, x^1, \ldots, x^n)}{dt} = X^i \circ \phi(\theta^1, \ldots, \theta^n)$$

with initial conditions $\theta^i(0, x^1, \dots, x^n) = (x^1, \dots, x^n)$.

It is a well-known fact from the theory of ordinary differential equations that a system of the above form has a unique solution in some neighbourhood that depends smoothly on the initial conditions.Thus, given a point $y \in M$, there is a neighbourhood U_y and a real number $\varepsilon_y > 0$ such that the equation

$$\frac{d\theta(t,x)}{dt} = X(\theta(t,x)) \qquad \theta(0,x) = x$$

has a unique solution for $x \in U_y$ and $|t| < \varepsilon_y$.

Now, let the vector field X vanish outside of some compact set K. We can find points $y_1, \dots, y_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} U_{y_i}$$

Let $\varepsilon = \inf\{\varepsilon_{y_1}, \dots, \varepsilon_{y_n}\}$. Then the equation

$$\frac{d\theta(t,x)}{dt} = X(\theta(t,x)) \qquad \theta(0,x) = x$$

has a unique smooth solution for $x \in K$ and $|t| < \varepsilon$.

For a point $x \notin K$ the vector $X(x)$ is zero, so the function $\theta(t,x) = x$ is a solution. Thus we have a smooth map $\theta\colon (-\varepsilon, \varepsilon) \times M \to M$ satisfying the given differential equation.

Suppose that $|s| < \varepsilon$, $|t| < \varepsilon$, and $|s+t| < \varepsilon$. Then:

$$\frac{d\theta(s+t,x)}{ds} = X(\theta(s+t,x)) \qquad \frac{d\theta(x,\theta(t,x))}{ds} = X(\theta(s,\theta(t,x)))$$

At $s = 0$:

$$\left.\frac{d\theta(s+t,x)}{ds}\right|_{s=0} = X(\theta(t,x)) \qquad \left.\frac{d\theta(x,\theta(t,x))}{ds}\right|_{s=0} = X(\theta(t,x))$$

Thus the solutions to the above two differential equations are the same and we have the formula

$$\theta(s+t,x) = \theta(s,\theta(t,x))$$

All that remains is to extend the solution $\theta(t,x)$ to hold when $|t| \geq \varepsilon$. Let $t \in \mathbb{R}$. Write

$$t = \frac{k\varepsilon}{2} + r; \qquad k \in \mathbb{Z}, \qquad |r| < \frac{\varepsilon}{2}$$

If $k \geq 0$ define

$$\theta(t,-) = \underbrace{\theta(\frac{\varepsilon}{2},-) \circ \cdots \circ \theta(\frac{\varepsilon}{2},-)}_{k \text{ times}} \circ \theta(r,-)$$

and if $k < 0$ define

$$\theta(t,-) = \underbrace{\theta(-\frac{\varepsilon}{2},-) \circ \cdots \circ \theta(-\frac{\varepsilon}{2},-)}_{-k \text{ times}} \circ \theta(r,-)$$

Then we have defined a smooth function $\theta\colon \mathbb{R} \times M \to M$ such that:

- $\theta(0,x) = x$ for all $x \in M$
- $\theta(s+t,x) = \theta(s,\theta(t,x))$ for all $s,t \in \mathbb{R}$ and $x \in M$
- $$X(f)(x) = \lim_{h \to 0} \left[\frac{f(\theta(h,x)) - f(x)}{h} \right]$$

as required. □

Note that in the above we need the conditions that the manifold M is closed and the vector field X is compactly supported for the result to be true. Without these conditions, the result is not true.

Consider a smooth map $c: \mathbb{R} \to M$. In the course of the above proof we introduced the *velocity vector*

$$\frac{dc}{dt} \in T_{c(t)}M$$

Let (U,V,ϕ) be a chart, and write

$$c(t) = \phi(x^1(t), \ldots, x^n(t))$$

and

$$\frac{dc}{dt} = \frac{dx^1}{dt}\frac{\partial}{\partial x^1} + \cdots + \frac{dx^x}{dt}\frac{\partial}{\partial x^n}$$

Definition 5.19 Let X be a vector field on a manifold M. Then an *integral curve* for X is a curve $c: \mathbb{R} \to M$ such that

$$\frac{dc}{dt} = X(c(t))$$

for all $t \in \mathbb{R}$.

Proposition 5.20 *Let X be a compactly supported vector field on a closed manifold M. Consider a point $x \in M$. Then there is a unique integral curve $c: \mathbb{R} \to M$ for the field X such that $c(0) = x$.*

Proof: By theorem 5.18 there is a unique flow $\theta: \mathbb{R} \times M \to M$ with tangent vector field X. We can define an integral vector field X by the formula

$$c(t) = \theta(t,x)$$

Uniqueness of the integral curve c follows locally by the theory of ordinary differential equations. Global uniqueness therefore holds. □

If $\theta: \mathbb{R} \times M \to M$ is a flow on M, a *flux line* for θ is a curve of the form

$$t \mapsto \theta(t,x)$$

for some fixed point $x \in M$. Flux lines for a flow are the same thing as integral curves for the tangent vector field of a flow.

We conclude with a characterisation of integral curves.

Theorem 5.21 *Let $c: \mathbb{R} \to M$ be an integral curve. Then one of the following three possibilities hold:*

- *The map c is constant.*
- *The map c is an embedding of $\mathbb{R}$ in M.*
- *The map c is an immersion and there is a positive number $p > 0$ such that $c(t) = c(t+p)$ for all $t \in \mathbb{R}$*

Proof: Suppose there is a point $t_0 \in \mathbb{R}$ such that $dc/dt_0 = 0$. Write $c(t_0) = x_0$. Then the curve $c(t)$ satisfies the differential equation

$$\frac{dc}{dt} = X(c(t)) \qquad c(t_0) = x_0$$

The condition $dc/dt_0 = 0$ implies that the map c must be constant, and we have the first of the above possibilities.

If the map c is not constant, the above argument tells us that the map $c: \mathbb{R} \to M$ is an immersion. If the map c is injective, it is also an embedding, and we have the second of the above possibilities.

Thus if neither the first nor the second of the above possibilities hold, the map c is an immersion, and we can find points $t_0 \in \mathbb{R}$ and $p > 0$ such that $c(t_0) = c(t_0 + p)$. The map $t \mapsto c(t+p)$ thus satisfies the differential equation

$$\frac{dc(t+p)}{dt} = X(c(t+p)) \qquad c(t_0 + p) = c(t_0)$$

which is the same equation as that used to define the integral curve c.

Hence $c(t) = c(t+p)$ for all points $t \in \mathbb{R}$, and we are done. □

5.4 Tensors and Duals

Let V and W be real vector spaces. Suppose that V and W are finite-dimensional, with bases $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$ respectively. Then the *tensor product* $V \otimes W$ is the vector space with a basis made up of symbols, which we write $e_i \otimes f_j$, where $i = 1, \ldots, m,\ j = 1, \ldots, n$.

Thus, the space $V \otimes W$ has dimension mn. Given vectors

$$v = \alpha^1 e_1 + \cdots + \alpha^n e_n,\ w = \beta^1 f_1 + \cdots + \beta^n f_n$$

we define the *elementary tensor*

$$v \otimes w = \sum_{i,j=1}^{m,n} \alpha^i \beta^j e_i \otimes f_j.$$

It is easy to check that these elementary tensors satisfy the relations:

- $\alpha(v \otimes w) = (\alpha v) \otimes w = v \otimes (\alpha w)$ where $\alpha \in \mathbb{R}$, $v \in V$, $w \in W$.
- $(v + v') \otimes w = v \otimes w + v' \otimes w$ where $v, v' \in V$ and $w \in W$.
- $v \otimes (w + w') = v \otimes w + v \otimes w'$.

We have a natural bilinear map $\sigma: V \times W \to V \otimes W$ defined by the formula $\sigma(v, w) = v \otimes w$. The following result is called the *universal property* of the tensor product.

Theorem 5.22 *Let $T: V \times W \to A$ be a bilinear map, where A is another vector space. Then there is a unique linear map $\tilde{T}: V \otimes W \to A$ such that $T = \tilde{T}\sigma$.*

Proof: Define $\tilde{T}: V \otimes W \to A$ by the formula

$$\tilde{T}\left(\sum_{i,j=1}^{m,n} \gamma^{ij} e_i \otimes f_j\right) = \sum_{i,j=1}^{m,n} T(e_i, e_j).$$

Let

$$v = \sum_{i=1}^{m} \alpha^i e_i \qquad w = \sum_{j=1}^{n} \beta^j f_j$$

Then

$$\tilde{T}(v \otimes w) = \sum_{i,j=1}^{m,n} \alpha^i \beta^j T(e_i, e_j) = T(v, w)$$

so $T = \tilde{T}\sigma$.

Suppose $T': V \otimes W \to A$ is a linear map where $T = T'\sigma$. Then

$$T'(e_i \otimes f_j) = T(e_i, f_j).$$

Thus T' agrees with $\tilde{T}$ on each basis element $e_i \otimes f_j$. By linearity, we have $T' = \tilde{T}$, and uniqueness follows. □

Corollary 5.23 *Suppose we have a vector space X equipped with a bilinear map $\tau: V \times W \to X$ where for any bilinear map $T: V \times W \to A$ there is a unique linear map $\tilde{T}: X \to A$ such that $T = \tilde{T}\tau$.*

Then $X \equiv V \otimes W$. □

Definition 5.24 Let E and F be vector bundles over a space X. Then we define the vector bundle $E \otimes F$ to be the union of the fibres $(E \otimes F)_x = E_x \otimes F_x$.

The topology is defined by defining an atlas of local trivialisations as follows; let $\{(U_\lambda, \psi_\lambda^{(E)}) \mid \lambda \in \Lambda\}$ and $\{(U_\lambda, \psi_\lambda^{(F)}) \mid \lambda \in \Lambda\}$ be atlases of local trivialisations for the bundles E and F respectively. Then we define local trivialisations $h_\lambda: (E \otimes F)_{U_\lambda} \to U_\lambda \times (\mathbb{R}^m \otimes \mathbb{R}^n)$ by the formula

$$h_\lambda(e \otimes f) = \psi_\lambda^{(E)}(e) \otimes \psi_\lambda^{(F)}(f)$$

where $x \in U_\lambda$ and $v \in \mathbb{R}^m$.

If E and F are smooth vector bundles, then so is $E \otimes F$.

We now turn to the second of our constructions. Let V be a real vector space. We define the *dual space* V^*, to be the set $Hom(V, \mathbb{R})$ of all linear maps $V \to \mathbb{R}$. This set is a vector space, with addition and scalar multiplication defined by writing

$$(f + g)(v) = f(v) + g(v) \qquad f, g \in V^*, v \in V$$

and

$$(\alpha f)(v) = \alpha f(v) \qquad \alpha \in \mathbb{R}, f \in V$$

respectively.

Note that given a linear map $T: V \to W$, we have a *dual linear map* $T^*: W^* \to V^*$ defined by writing $T(f)(v) = f(T(v))$ where $f \in W^*$ and $v \in V$. If the map T is an isomorphism, then so is the dual map T^*.

Suppose that V is finite-dimensional, with basis $\{e_1, \dots, e_n\}$. Define $e_i^* \in V^*$ by

$$e_i^*(\alpha^1 e_1 + \cdots + \alpha^n e_n) = \alpha_i.$$

Then it is easy to check that $\{e_1^*, \dots, e_n^*\}$ is a basis for V^*. We call it the *dual basis.*

Although this construction proves that V and V^* are isomorphic when V is finite-dimensional, the isomorphism depends on the chosen basis.

Proposition 5.25 *Let V be finite-dimensional. Then we have a basis-independent isomorphism $\tau: V \to (V^*)^*$.*

Proof: We can define a linear map $\tau: V \to (V^*)^*$ by

$$\tau(v)(f) = f(v) \qquad f \in V^*, v \in V.$$

If $\tau(v) = 0$, then $f(v) = 0$ for all $f \in Hom(V, \mathbb{R})$. But this clearly implies $v = 0$. So τ is injective. By the above

$$\dim(V) = \dim(V^*) = \dim((V^*)^*)$$

so the linear injection τ is an isomorphism. □

Definition 5.26 Let E and F be vector bundles over a space X. Then we define the vector bundle E^* to be the union of the fibres $(E_x)^*$.

The topology is defined by defining an atlas of local trivialisations as follows; let $\{(U_\lambda, \psi_\lambda) \mid \lambda \in \Lambda\}$ be an atlas of local trivialisations for the bundle E. Then we define local trivialisations $h_\lambda: E^*_{U_\lambda} \to U_\lambda \times (\mathbb{R}^n)^*$ by the formula

$$h_\lambda(f) = (\psi_\lambda^*)^{-1}(f) \qquad f \in E_x^*, \ x \in U_\lambda$$

If E is a smooth vector bundle, then so is E^*.

Now, let V and W be real vector spaces, and let $Hom(V, W)$ be the set of all linear maps $V \to W$. This set is a vector space, with addition and scalar multiplication defined by writing

$$(S + T)(v) = S(v) + T(v) \qquad S, T \in Hom(V, W), v \in V$$

and

$$(\alpha T)(v) = \alpha T(v) \qquad \alpha \in \mathbb{R}, T \in Hom(V, W)$$

respectively.

If $\dim(V) = m$ and $\dim(W) = n$, then $\dim Hom(V, W) = mn$.

Lemma 5.27 *Let V and W be finite-dimensional real vector spaces. Then we have an isomorphism $Hom(V, W) \cong V^* \otimes W$.*

Proof: Define $\gamma\colon V^* \otimes W \to Hom(V, W)$ by the formula

$$\gamma(f \otimes w)(v) = f(v)w \qquad f \in V^*,\ v \in V,\ w \in W.$$

Then γ is a linear map. If $\gamma(f \otimes w) = 0$, then $f(v)w = 0$ for all $v \in V$. This means either $f = 0$ or $w = 0$; in either case, $f \otimes w = 0$, and the map γ is injective.

Now, $dim(V^* \otimes W) = \dim(V)\dim(W) = \dim Hom(V, W)$ so the map γ is also surjective, and we are done. □

This suggests the following.

Definition 5.28 Let E and F be vector bundles over a space X. Then we define $Hom(E, F)$ to be the set of bundle maps from E to F. We have fibres $Hom(E, F)_x = Hom(E_x, F_x) \equiv (E_x)^* \otimes F_x$.

The topology on $Hom(E, F)$ is defined in such a way as the natural bijection $Hom(E, F) \to E^* \otimes F$ is a bundle isomorphism.

We can of course also define the topology on $Hom(E, F)$ through an atlas of local trivialisations, in the same way as for duals and tensor products. If E and F are smooth vector bundles, then so is $Hom(E, F)$.

5.5 Tensor Fields

A *tensor field* is a generalisation of a vector field to several variables. Tensor fields frequently arise in differential geometry.

Definition 5.29 Let M be a smooth manifold. Then we define the *cotangent bundle* of M to be the dual of the tangent bundle. A section of the cotangent bundle is called a *covector field*.

We denote the cotangent bundle of the manifold M by the symbol $T^\star M$. The fibre at the point $x \in M$ is called the *space of cotangent vectors* and denoted by the symbol $T^\star_x M$.

Let (U, V, ϕ) be a chart, and $\phi(x^1, \ldots, x^n) = x \in V$. Then the tangent space $T^\star_x M$ has a basis

$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$$

The cotangent space has a dual basis,

$$dx^1, \ldots, dx^n$$

where

$$dx^i \left(\frac{\partial}{\partial x^j} \right) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Let $(\tilde{U}, \tilde{V}, \tilde{\phi})$ be another chart, where $\tilde{\phi}(\tilde{x}^1, \ldots, \tilde{x}^n) = x$

Then the cotangent vector

$$\alpha_1 dx^1 + \cdots + \alpha_n dx^n$$

becomes

$$\tilde{\alpha}_1 d\tilde{x}^1 + \cdots + \tilde{\alpha}_n d\tilde{x}^n$$

where

$$\tilde{\alpha}_j = \sum_i \alpha_i \frac{\partial x^i}{\partial \tilde{x}^j}$$

by the rule for changing coordinates for tangent vectors.

Example 5.30 Let $f\colon M \to \mathbb{R}$ be a smooth function. Then we define a covector field $df \in \Gamma(T^\star M)$ by writing in local coordinates:

$$df = \frac{\partial f}{\partial x^1} dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^n$$

Looking at bases in local coordinates yields the following result.

Proposition 5.31 *Every covector field has the form df for some smooth function f.* □

Definition 5.32 A *tensor field* of *type* (p, q) is a section of the tensor product:

$$\underbrace{TM \otimes \cdots \otimes TM}_{p \text{ times}} \otimes \underbrace{T^\star M \otimes \cdots \otimes T^\star M}_{q \text{ times}}$$

As special cases, note that a vector field is a tensor of type $(1, 0)$, and a covector field is a tensor of type $(0, 1)$. A tensor field of type $(0, 0)$ is simply a smooth function $f\colon M \to \mathbb{R}$. We write $\Gamma^p_q(M)$ to denote the set of tensor fields of type (p, q).

Let (U, V, ϕ) be a chart, and let $(x^1, \ldots, x^n) \in U$. Then a tensor, A, of type (p, q) takes the form:

$$\sum_{i_1,\ldots,i_p,j_1,\ldots,j_q} A^{i_1\cdots i_p}_{j_1\cdots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q}$$

If we pick a different chart $(\tilde{U}, \tilde{V}, \tilde{\phi})$ where $p\tilde{h}i(\tilde{x}^1, \ldots, \tilde{x}^n) = \phi(x^1, \ldots, x^n)$, the tensor A is expressed by the formula

$$\sum_{i_1,\ldots,i_p,j_1,\ldots,j_q} \tilde{A}^{i_1\cdots i_p}_{j_1\cdots j_q} \frac{\partial}{\partial \tilde{x}^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial \tilde{x}^{i_p}} \otimes d\tilde{x}^{j_1} \otimes \cdots \otimes d\tilde{x}^{j_q}$$

where:

$$\tilde{A}^{i_1\cdots i_p}_{j_1\cdots j_q} = \sum_{k_1,\ldots,k_p,l_1,\ldots,l_q} A^{k_1\cdots k_p}_{l_1\cdots l_q} \frac{\partial \tilde{x}^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial \tilde{x}^{i_p}}{\partial x^{k_p}} \frac{\partial x^{l_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{l_q}}{\partial \tilde{x}^{j_q}}$$

Classically, a tensor field of type (p, q) is defined to be a set of functions $A^{i_1\cdots i_p}_{j_1\cdots j_q}$ on the coordinates defined by the charts that transforms according to the above rule. The modern approach in mathematics is to avoid such messy implicit descriptions of tensors wherever possible.

However, if such a definition is forced upon us, it is worth bearing in mind that the above formula is not quite as nasty as it at first appears. Note that any summation is over pairs of indices, one higher and one lower.[2] The *Einstein*

[2]Actually, we have tried to keep to this convention throughout these notes.

summation convention omits the summation sign from such expressions; we assume that any pair of indices, one higher and one lower, is summed over. According to this convention, the above expression takes the form

$$\tilde{A}^{i_1\cdots i_p}_{j_1\cdots j_q} = A^{k_1\cdots k_p}_{l_1\cdots l_q} \frac{\partial \tilde{x}^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial \tilde{x}^{i_p}}{\partial x^{k_p}} \frac{\partial x^{l_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{l_q}}{\partial \tilde{x}^{j_q}}$$

Actually, we will not use the Einstein summation convention in these notes, but it is worth bearing in mind for calculations done in private. Many other references, especially in mathematical physics, do use the Einstein summation convention, so we need to at least be aware of its existence.

The following result is extremely helpful when defining tensors, especially when we seek to avoid a definition based on local coordinates.

Example 5.33 The *Kronecker delta function*, δ, is the tensor of type $(1, 1)$ defined by $\delta(x) = 1 \in Hom(T_xM, T_xM)$ for all $x \in M$.

In coordinates $(x^1, \ldots, x^n)$ from some chart it takes the form

$$\delta = \sum_i \frac{\partial}{\partial x^i} \otimes dx^i$$

or

$$\delta = \sum_{i,j} \delta^i_j \frac{\partial}{\partial x^i} \otimes dx^j$$

where

$$\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

By lemma 5.27, a tensor field of type (p, q) is the same thing as a smooth section of the bundle

$$Hom\left(\underbrace{TM \otimes \cdots \otimes TM}_{q \text{ times}}, \underbrace{TM \otimes \cdots \otimes TM}_{p \text{ times}}\right)$$

Example 5.34 Suppose that we have a metric on the tangent bundle TM (see definition 5.5). Then we have a tensor g of type $(0, 2)$ defined by the formula

$$g(X \otimes Y)(x) = \langle X(x), Y(x) \rangle$$

With respect to coordinates $(x^1, \ldots, x^n)$ defined by some chart it takes the form

$$g = \sum_{i,j} g_{ij} dx^i \otimes dx^j$$

where $g_{ij} = \langle \partial/\partial x^i, \partial/\partial x^j \rangle$.

6 Riemannian Geometry

6.1 Riemannian Metrics

Definition 6.1 A *Riemannian metric* on a smooth manifold M is a metric on the tangent bundle TM. A smooth manifold equipped with a Riemannian metric is called a *Riemannian manifold*.

Thus a Riemannian manifold has a collection of inner products $\langle -,-\rangle : T_xM \otimes T_xM \to \mathbb{R}$ such that the map:

$$x \mapsto \langle u(x), v(x)\rangle$$

is smooth for all smooth sections $u, v \in \Gamma^\infty(TM)$.

By theorem 5.6 any smooth manifold can be equipped with a Riemannian metric.

As we discussed in example 5.34, a metric on a manifold M defines a tensor field g, of type $(2,0)$, by the formula

$$g(X,Y)(x) = \langle X(x), Y(x)\rangle$$

We also have a converse. Let $g : \Gamma^\infty(TM) \otimes_{C^\infty(M)} \Gamma^\infty(TM) \to \Gamma^\infty(TM)$ be a tensor field of type $(0,2)$. Call g *symmetric* if $g(X,Y) = g(Y,X)$ for all vector fields X and Y, and *positive definite* if $g(X,X)(x) > 0$ for any vector field X such that $X(x) \neq 0$. We have the following obvious result.

Proposition 6.2 *Let g be a symmetric and non-degenerate tensor field of type $(0,2)$ on a manifold M. Then we can define a Riemannian metric on the manifold M by the formula*

$$\langle X(x), Y(x)\rangle = g(X,Y)(x)$$

□

We also refer to the tensor field g as the *metric* on M. For a chart (U,V,ϕ) where $(x^1,\dots,x^n) \in U$, let us write

$$g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle$$

Then for vector fields

$$X = \sum_i X^i \frac{\partial}{\partial x^i} \qquad Y = \sum_j Y^j \frac{\partial}{\partial x^j}$$

we have the inner product

$$\langle X, Y\rangle = \sum_{i,j} g_{ij} X^i Y^j$$

Proposition 6.3 *The matrix of smooth functions (g_{ij}) is invertible.*

Proof: The desired result is now a statement in elementary linear algebra about finite-dimensional inner product spaces; if V is a finite-dimensional inner product space, with basis $\{e_1,\dots,e_n\}$, then the matrix with coefficients A_{ij} defined by the formula

$$A_{ij} = \langle e_i, e_j\rangle$$

is invertible.

Now apply this to the matrix $(g_{ij}(x))$. □

We write (g^{ij}) to denote the inverse of the matrix of smooth functions (g_{ij}). It is itself a matrix of smooth functions. The following is straightforward.

Proposition 6.4 *We can define a tensor, g^{-1}, of type $(2,0)$ by writing*

$$g^{-1} = \sum_{i,j} g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$$

with respect to a chart. □

Example 6.5 Consider Euclidean space, $\mathbb{R}^n$, with the usual coordinate system $(x^1, \ldots, x^n)$. We can define a metric on $\mathbb{R}^n$ by writing

$$\langle X, Y \rangle = \sum_i X^i Y^i$$

for vector fields

$$X = \sum_i X^i \frac{\partial}{\partial x^i} \qquad Y = \sum_j Y^j \frac{\partial}{\partial x^j}$$

The following result follows from the definition of an immersion.

Proposition 6.6 *Let $i\colon M \to N$ be an immersion, and suppose we have a metric $\langle -, - \rangle_N$ on the manifold N. Then we have a metric on the manifold M defined by the formula*

$$\langle X, Y \rangle_M = \langle f_\star(X), f_\star(Y) \rangle_N$$

□

In particular, a manifold embedded in Euclidean space inherits a metric based on the embedding.

Definition 6.7 Let M and N be Riemannian manifolds. Then a smooth map $f\colon M \to N$ is called an *isometry* if:

$$\langle f_\star(X), f_\star(Y) \rangle_M = \langle X, Y \rangle_N$$

for all vector fields X and Y over the manifold M.

Isometries between Riemannian manifolds are the 'distance-preserving' maps. We call two Riemannian manifolds M and N *isometric* if there is a diffeomorphism $f\colon M \to N$ that is also an isometry.

For example, if $i\colon M \to N$ is an immersion, N is a Riemannian manifold, and we define a metric on the manifold M as in proposition 6.6, then the immersion i is an isometry. Conversely, we have the following obvious result.

Proposition 6.8 *Let $i\colon M \to N$ be an isometry between Riemannian manifolds. Then i is an immersion.* □

6.2 Connections

Let M be a smooth manifold, and let $X \in \Gamma^\infty(TM)$ be a vector field. Recall from definition 5.9 that for any smooth function $f \in C^\infty(M)$ we can define a smooth function $X(f) \in \Gamma(TM)$ by writing

$$X(f)(x) = (f \circ \gamma)'(0) \quad \gamma\colon (-\delta, \delta) \to M,\ \gamma(0) = x,\ \gamma'(0) = X(x).$$

A *connection* on a smooth vector bundle E over M is a piece of structure that enables us to perform a similar operation of differentiation of sections of the bundle E.

Definition 6.9 A *connection* on a smooth vector bundle E is a bilinear map $\nabla\colon \Gamma^\infty(TM) \times \Gamma^\infty(E) \to \Gamma^\infty(E)$, written $(X, u) \mapsto \nabla_X u$, such that:

- $\nabla_{fX} u = f \nabla_X u$
- $\nabla_X(fu) = X(f)u + f\nabla_X u$

for all vector fields $X \in \Gamma^\infty(TM)$, smooth sections $u \in \Gamma^\infty(E)$, and smooth functions $f \in C^\infty(M)$.

For most of this section we concentrate on connections on the tangent bundle, TM.

Example 6.10 Consider Euclidean space, $\mathbb{R}^n$, with the usual coordinate system $(x^1, \ldots, x^n)$. We can define a connection on $T\mathbb{R}^n$ by writing

$$\nabla_X Y = \sum_{i,j} X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j}$$

for vector fields

$$X = \sum_i X^i \frac{\partial}{\partial x^i} \qquad Y = \sum_j Y^j \frac{\partial}{\partial x^j}$$

In what follows, we will need to do some calculations in terms of a chart (U, V, ϕ) and $(x^1, \ldots, x^n) \in U$. To streamline the process, let us make the abbreviations:

$$\partial_i = \frac{\partial}{\partial x^i} \qquad \nabla_i = \nabla_{\partial_i}$$

Definition 6.11 We define the *Christoffel symbols*, Γ_{ij}^k, of a connection ∇ on the tangent bundle TM by writing:

$$\nabla_i \partial_j = \sum_k \Gamma_{ij}^k \partial_k$$

Note that despite our notation, the Christoffel symbols *do not* define tensor fields.

Proposition 6.12 *The Christoffel symbols completely determine the connection ∇.*

Proof: Let X and Y be vector fields. Write:

$$X = \sum_i X^i \partial_i \qquad Y = \sum_j X^j \partial_j$$

Then:

$$\begin{aligned} \nabla_X Y &= \textstyle\sum_i X^i \nabla_i Y \\ &= \textstyle\sum_{i,j} X^i((\partial_i Y^j)\partial_j + Y^j \nabla_i \partial_j) \\ &= \textstyle\sum_{i,j,k} X^i((\partial_i Y^j)\partial_j + Y^j \Gamma_{ij}^k \partial_k) \end{aligned}$$

and we are done. □

Recall that for vector fields X and Y we define the *Lie bracket* $[X, Y]$ by the formula:

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

where $f \in C^\infty(M)$.

Definition 6.13 Let ∇ be a connection on TM. Then we call the connection ∇ *symmetric* if the formula:

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

is satisfied.

It is not hard to check that the connection ∇ is symmetric if and only if the Christoffel symbols satisfy the identity

$$\Gamma_{ij}^k = \Gamma_{ji}^k$$

Definition 6.14 Let M be a Riemannian manifold. Then a connection ∇ on TM is said to be *compatible with the metric* if the formula:

$$X(\langle Y, Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

is satisfied.

Actually the above definition makes sense for any bundle, E, equipped with a connection and a metric.

Theorem 6.15 *The Fundamental Theorem of Riemannian Geometry*
Let M be a Riemannian manifold. Then there is a unique symmetric connection on TM that is compatable with the metric.

Proof: Let ∇ be a symmetric connection on the manifold M that is compatable with the metric. Consider the Christoffel symbols Γ_{ij}^k. The compatability conditions give us an equation:

$$\partial_i g_{jk} = \sum_l \langle \nabla_i \partial_j, \partial_k\rangle + \langle \partial_j, \nabla_i \partial_k\rangle = \sum_l \Gamma_{ij}^l g_{lk} + \Gamma_{ik}^l g_{lj}$$

for every permutation of the set of indices $\{i, j, k\}$. By symmetry of the connection we can solve the above set of equations to yield the identity:

$$\sum_l \Gamma_{ij}^l g_{lk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$$

and so:

$$\Gamma^l_{ij} = \frac{1}{2}\sum_k (\partial_i g_j k + \partial_j g_{ik} - \partial_k g_{ij}) g^{kl}$$

Thus the Christoffel symbols, and therefore the connection, are determined by the metric. Conversely, if we define the Christoffel symbols by the above formula, we obtain a connection that is symmetric and compatable with the metric. □

The connection defined in the above theorem is called the *Levi-Cevita* connection of the Riemannian manifold M.

Example 6.16 Recall from example 6.10 that we can define a connection on Euclidean space, $\mathbb{R}^n$ by writing

$$\nabla_X Y = \sum_{i,j} X^i (\partial_i Y^j) \partial_j$$

for vector fields

$$X = \sum_i X^i \partial_i \qquad Y = \sum_j Y^j \partial_j$$

with respect to the standard coordinates $(x^1, \ldots, x^n)$.

Observe that:

$$\nabla_X Y - \nabla_Y X = \sum_{i,j} (X^i \partial_i Y^j - Y^i \partial_i X^j) \partial_j$$

which is just an expression for the Lie bracket $[X, Y]$ (see the proof of proposition 5.11). Thus the given connection is symmetric.

Let $\langle -, - \rangle$ be the metric defined on $\mathbb{R}^n$ in example 6.5. Consider vector fields

$$X = \sum_i X^i \partial_i \qquad Y = \sum_j Y^j \partial_j \qquad Z = \sum_k Z^k \partial_k$$

Then:

$$X(\langle Y, Z \rangle) = \sum_{i,j} ((X^i \partial_i Y^j) Z^j \partial_j + Y^j (X^i \partial_i Z^j) \partial_j) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

so the connection is compatible with the metric.

Hence the given connection is the Levi-Cevita connection for Euclidean space equipped with the standard metric.

Let $c: \mathbb{R} \to M$ be a smooth curve on a manifold M. A *vector field along* c, Y, consists of a tangent vector $Y_t \in T_{c(t)}(M)$ for each point $t \in \mathbb{R}$ such that for any smooth function $f \in C^\infty(M)$ the map

$$t \mapsto Y_t(f)$$

is smooth.

Example 6.17 The *velocity vector field*, dc/dt, of the curve c is a vector field along c.

Proposition 6.18 *Let M be a smooth manifold equipped with a connection ∇. Let Y be a vector field along a smooth curve $c: \mathbb{R} \to M$. Then there is a unique vector field DY/Dt characterised by the following properties.*

- *If Y and Z are both vector fields along c then*
$$\frac{D(Y+Z)}{Dt} = \frac{DY}{Dt} + \frac{DZ}{Dt}$$
- *For a smooth function $f: \mathbb{R} \to \mathbb{R}$ and a vector field Y along c we have the equation*
$$\frac{D(fY)}{Dt} = \frac{df}{dt}Y + f\frac{DY}{Dt}$$
- *Let X be a vector field on M such that $X(c(t)) = Y_t$. Then:*
$$\nabla_{dc/dt}X = \frac{DY}{Dt}$$

Proof: Let (U, V, ϕ) be a chart. Write $c(t) = \phi(x^1(t), \ldots, x^n(t))$ and

$$Y = \sum_i Y^i \partial_i$$

for some smooth functions Y^i defined on an open subset of $\mathbb{R}$. The stated conditions tell us that:

$$\frac{DY}{Dt} = \sum_i \left(\frac{dY^i}{dt}\partial_i + Y^i \nabla_{dc/dt}\partial_i \right)$$

so

$$\frac{DY}{Dt} = \sum_k \left(\frac{dY^k}{dt} + \sum_{i,j} \Gamma^k_{ij} \frac{dx^i}{dt} Y^j \right) \partial_k$$

by definition of the Christoffel symbols Γ^k_{ij}.

Conversely, the above formula defines a vector field DY/Dt along c that has the required properties. □

Definition 6.19 We call the vector field DY/Dt the *covariant derivative* of Y. The vector field V along c is called a *parallel vector field* if the covariant derivative is zero.

The usual methods of establishing the existence of unique solutions to first order differential equations give us the following.

Proposition 6.20 *Let $c: \mathbb{R} \to M$ be a smooth curve. Let $Y_0 \in T_{c(0)}M$. Then there is a unique parallel vector field Y_t along c that is equal to the vector Y_0 when $t = 0$.* □

We say that the vector $Y_t \in T_{c(t)}M$ is obtained from Y_0 by *parallel transport* along c.

Definition 6.21 Let M be a Riemannian manifold equipped with some connection. Let us say that *parallel transprort preserves inner products* if for any curve c and any two parallel vector fields Y and Z along c the product $\langle Y_t, Z_t\rangle$ is constant.

The following is straightforward.

Proposition 6.22 *The connection on the manifold M is compatible with the metric if and only if parallel translation preserves inner products.* □

We end this section by noting that we can formulate a two-dimensional version of proposition 6.18; the proof is virtually identical.

Definition 6.23 Let M be a smooth manifold equipped with a connection ∇. Let $\sigma: \mathbb{R}^2 \to M$ be a smooth map (we could call σ a *smooth surface* in M).

A *vector field along* σ, Y, consists of a tangent vector $Y_{x,y} \in T_{\sigma(x,y)}(M)$ for each point $(x, y) \in \mathbb{R}^2$ such that for any smooth function $f \in C^\infty(M)$ the map

$$(x, y) \mapsto Y_(x, y)(f)$$

is smooth.

Proposition 6.24 *There are unique vector fields $DY/\partial x$ and $DY/\partial y$ characterised by the following properties.*

- *If Y and Z are vector fields along σ then*

$$\frac{D(Y+Z)}{\partial x} = \frac{DY}{\partial x} + \frac{DZ}{\partial x} \qquad \frac{D(Y+Z)}{\partial y} = \frac{DY}{\partial y} + \frac{DZ}{\partial y}$$

- *For a smooth function $f: \mathbb{R}^2 \to \mathbb{R}$ and a vector field V along σ we have the equations*

$$\frac{D(fY)}{\partial x} = \frac{\partial f}{\partial x}Y + f\frac{DY}{\partial x} \qquad \frac{D(fY)}{\partial y} = \frac{\partial f}{\partial y}Y + f\frac{DY}{\partial y}$$

- *Let X be a vector field on M such that $X(\sigma(x, y)) = Y_{x,y}$. Then:*

$$\nabla_{\partial c \partial x} X = \frac{DY}{\partial x} \qquad \nabla_{\partial c \partial y} X = \frac{DY}{\partial y}$$

□

6.3 Geodesics

Let M be a connected Riemannian manifold, equipped with the Levi-Cevita connection.

Definition 6.25 Let $I \subseteq \mathbb{R}$ be some interval. A smooth curve $\gamma: I \to M$ is called a *geodesic* if the *acceleration vector field* $D/Dt(\gamma'(t))$ is zero for all $t \in I$.

Note that if γ is a geodesic, by definition the velocity vector field $\gamma'(t)$ is parallel. Observe that:

$$\frac{d}{dt}\langle\gamma'(t),\gamma'(t)\rangle = 2\langle\frac{D}{Dt}\left(\gamma'(t)\right), \frac{d\gamma}{dt}\rangle = 0$$

Hence the norm

$$\|\gamma'(t)\| = \langle\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle^{\frac{1}{2}}$$

is constant.

Definition 6.26 Let $c\colon [a,b] \to M$ be a piecewise-smooth curve. We define the *arclength* of c by the formula

$$\mathit{Length}(c) = \int_a^b \|c'(t)\| \; dt$$

The following result is obvious.

Proposition 6.27 *Let $c\colon [a,b] \to M$ be a piecewise-smooth curve, and let $f\colon M \to N$ be an isometry between Riemannian manifolds. Then:*

$$\mathit{Length}(c) = \mathit{Length}(F \circ c)$$

□

We will show in this section that geodesics are precisely those curves which locally minimise arclength. It is easy to see that if c' is a reparametrisation of a curve c, then $\mathit{Length}(c') = \mathit{Length}(c)$.

Consider a chart (U,V,ϕ) where $(x^1,\ldots,x^n) \in U$. Write $\gamma(t) = \phi(x^1(t),\ldots,x^n(t))$. Then, as in the proof of proposition 6.18

$$\frac{dY}{dt} = \sum_k \left(\frac{DY^k}{dt} + \sum_{i,j} \Gamma^k_{i,j} \frac{dx^i}{dt} Y^j \right) \partial_k$$

for any vector field Y along γ. In particular, the geodesic equation becomes the system of equations

$$\frac{d^2x^k}{dt^2} + \sum_{i,j} \Gamma^k_{i,j} \frac{dx^i}{dt}\frac{dx^j}{dt}$$

Proposition 6.28 *Let $x_0 \in M$. Then there is an open neighbourhood $U \ni x_0$ and a real number $\varepsilon > 0$ such that for each point $x \in U$ and tangent vector $V \in T_xM$ such that $\|V\| < \varepsilon$ there is a unique geodesic $\gamma_V\colon (-2,2) \to M$ such that*

$$\gamma_V(0) = x \qquad \frac{d\gamma_V}{dt}(0) = V$$

Moreover, the function

$$(V,t) \mapsto \gamma_V(t)$$

is smooth.

Proof: By choosing a suitable chart. the statement *almost* follows immediately from the theory of existence and uniqueness of solutions to ordinary differential equations. To be precise, there is an open neighbourhood $U \ni x_0$ and real numbers $\varepsilon_1, \varepsilon_2 > 0$ such that for every tangent vector $V \in T_xM$ with $\|V\| < \varepsilon_1$ there is a unique geodesic $\gamma_V \colon (-2\varepsilon_2, 2\varepsilon_2) \to M$ such that

$$\gamma_V(0) = x \qquad \frac{d\gamma_V}{dt}(0) = V$$

Further, the required smoothness condition holds.

Let $\varepsilon = \varepsilon_1\varepsilon_2$. If $\|V\| < \varepsilon$ and $|t| < 2$ note that $\|V/\varepsilon_2\| < \varepsilon_1$ and $|\varepsilon_2 t| < 2\varepsilon_2$. Thus we can define our geodesic by the formula

$$\gamma_V(t)\ \gamma_{V/\varepsilon_1}(\varepsilon_2 t)$$

□

Definition 6.29 Let $x \in M$ and let $V \in T_xM$. Suppose there exists a unique geodesic $\gamma_V \colon [0,1] \to M$ such that

$$\gamma_V(0) = x \qquad \frac{d\gamma_V}{dt}(0) = V$$

Then we define the *exponential function* of the vector V by writing

$$\exp_x(V) = \gamma_V(1)$$

We also write $\exp(x, V) = \gamma_V(1)$. Consider a point $x \in M$, and let $0_x \in T_xM$ be the zero vector. Proposition 6.28 tells us that there is an open neighbourhood $U \ni 0_x$ in the manifold TM such that we have a well-defined smooth exponential function

$$\exp \colon U \to M$$

Given a tangent vector $V \in T_xM$ we can describe the unique geodesic γ_V such that

$$\gamma_V(0) = x \qquad \frac{d\gamma_V}{dt}(0) = V$$

by the equation

$$\gamma_V(t) = \exp_x(tV)$$

for all sufficiently small t.

Proposition 6.30 *The arclength of the geodesic*

$$\gamma_V(t) = \exp_x(tV) \qquad t \in [0,1]$$

is equal to $\|V\|$.

Proof: The path γ_V is the unique geodesic such that

$$\gamma_V(0) = x \qquad \frac{d\gamma_V}{dt}(0) = V$$

Since γ_V is a geodesic, the norm $\|\gamma_V'(t)\|$ is constant. Hence $\|\gamma_V'(t)\| = V$ for all points $t \in [0,1]$. The desired result follows by definition of arclength. □

Theorem 6.31 *For each point $x_0 \in M$ there is a neighbourhood $W \ni x$ and a real number $\varepsilon > 0$ such that:*

- *Any two points of W are joined by a unique geodesic with arclength less than ε.*
- *Let*
$$t \mapsto \exp_x(tV) \qquad t \in [0,1]$$
be the unique geodesic of arclength less than ε joining two points $x, y \in W$. Then the map $M \times M \to TM$ defined by the formula $(x,y) \mapsto V$ is smooth.
- *For each point $x \in W$ the map*
$$\exp_x \colon D(0,\varepsilon) \to M$$
maps the open ε-ball in T_xM diffeomorphically onto an open set $U_x \supseteq W$.

Proof: Let $0_x \in T_{x_0}M$ be the zero tangent vector to x_0. By proposition 6.28 we have a well-defined smooth map

$$F \colon U_0 \to M \times M; \qquad F(V) = (x, \exp_x(V))$$

for all points $V \in T_xM$ belonging to some neighbourhood, U_0, of the point $0_{x_0} \in TM$.

Local coordinates $(x^1, \dots, x^n)$ on the manifold M give us local coordinates $(x^1, \dots, x^n, \alpha_1, \dots, \alpha_n)$ on the tangent bundle TM; the coordinates α^i are defined by writing a given tangent vector in the form

$$\alpha^1 \partial_1 + \cdots + \alpha^n \partial_n$$

Let us write $(x^1, \dots, x^n, y^1, \dots, y^n)$ to denote the local coordinates on the product manifold $M \times M$ induced by the local coordinated $(x^1, \dots, x^n)$ on M. Observe that:

$$F_\star\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i} \qquad F_\star\left(\frac{\partial}{\partial \alpha^i}\right) = \frac{\partial}{\partial y^i}$$

at the point $(x_0, 0)$.

Thus the differential of the map F at the point $(x_0, 0)$ has the form $\begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$ and so is invertible. By the inverse function theorem the map F is a diffeomorphism from some neighbourhood $V' \ni 0_{x_0}$ onto a neighbourhood $V'' \ni (x_0, x_0)$.

Let U' be a neighbourhood of the point $x_0 \in M$ such that V' contains all tangent vectors of the form $V \in T_xM$ where $x \in U'$ and $\|V\| < \varepsilon$. Let $W \ni x_0$ be a neighbourhood such that $F[U'] \supseteq W \times W$.

Then by definition of the map F the required properties of the neighbourhood W hold. □

We now introduce some terminology.

Definition 6.32 Consider a point $x \in M$. The largest real number $R > 0$ such that the exponential map

$$\exp_x \colon D(0,R) \to M$$

maps the open r-ball in T_xM diffeomorphically onto an open set $U_x \subseteq M$ is called the *injectivity radius* of M at the point x. If $\varepsilon \leq R$, the image $N(x, \varepsilon) = \exp_x[D(0, \varepsilon)]$ is termed a *geodesic neighbourhood* of the point x, of *radius* ε.

Thus, theorem 6.31 tells us that every point $x \in M$ has a non-zero injectivity radius, and so contains geodesic neighbourhoods.

It is possible for the injectivity radius to be infinite; for example, at every point $x \in \mathbb{R}^n$ the injectivity radius is infinite.

Theorem 6.33 *Let $x \in M$, and let U be a geodesic neighbourhood of x of radius ε. Let $\gamma\colon [0,1] \to M$ be a geodesic of length less than ε joining the point x to another point $y \in U$. Let $\omega\colon [0,1] \to M$ be a piecewise smooth path from x to y. Then $Length(\gamma) \leq Length(\omega)$.*

Further, equality holds if and only if the path ω is a reparametrisation of the path γ.

Thus, at least within geodesic neighbourhoods, geodesics are the unique arclength-minimising paths between points.

Before we prove the above theorem, we need to do some technical work. The first of the technical results we need is sometimes called *Gauss' lemma*.

Lemma 6.34 *Let U be a geodesic neighbourhood of a point $x \in M$. Then the geodesics through the point x in the set U are all perpendicular to the hypersurfaces*

$$\{\exp_x(V) \mid \|V\| = \textit{constant}\}$$

Proof: Let ε be the radius of the given geodesic neighbourhood U. Let $t \mapsto V(t)$ be a curve in the tangent space T_xM such that $\|V(t)\| = 1$. Suppose that $0 < r_0 < \varepsilon$. We must prove that the correponding curves

$$t \mapsto \exp_x(r_0 V(t)) \qquad r \mapsto \exp_x(rV(t_0))$$

are orthogonal.

Hence for the parametrised surface defined by the equation

$$f(r,t) = \exp_x(rV(t)) \qquad 0 \leq r < \varepsilon$$

we must prove that

$$\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \rangle = 0$$

for all r and t. A simple calculation yields the formula:

$$\frac{\partial}{\partial r}\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \rangle = \langle \frac{D}{\partial r}\left(\frac{\partial f}{\partial r}\right), \frac{\partial f}{\partial t} \rangle + \langle \frac{\partial f}{\partial r}, \frac{D}{\partial r}\left(\frac{\partial f}{\partial t}\right) \rangle$$

Since the curves $r \mapsto \exp_x(rV(t))$ are geodesics, the first expression on the right is zero. The second expression is equal to:

$$\langle \frac{\partial f}{\partial r}, \frac{D}{\partial t}\left(\frac{\partial f}{\partial r}\right) \rangle = \frac{1}{2}\frac{\partial}{\partial t}\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial r} \rangle = 0$$

since the norm $\|\partial f/\partial r\| = \|V(t)\|$ is constant.

Thus the quantity $\langle \partial f/\partial r, \partial f/\partial t\rangle$ does not depend on r. But for $r = 0$:

$$f(0,t) = \exp_x(0) = x$$

so $\partial f/\partial t(0,t) = 0$. Therefore

$$\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t}\rangle = 0$$

for all points $r \in [0, \varepsilon)$ and $t \in \mathbb{R}$ and we are done. □

The hypersurface

$$S(x,C) = \{\exp_x(V) \mid \|V\| = C\}$$

is called a *spherical shell* around x of radius C.

Consider a piecewise-smooth curve $\omega\colon [a,b] \to U\backslash\{x\}$, where I is a geodesic neighbourhood of the point $x \in M$ with radius ε. Each point $\omega(t)$ can be written uniquely in the form

$$\exp_x(r(t)V(t)) \qquad 0 < r(t) < \varepsilon \qquad \|V(t)\| = 1$$

Lemma 6.35 *We have the inequality*

$$Length(\omega) \geq |r(b) - r(a)|$$

Equality holds if and only if the function $r(t)$ is monotone and the function $V(t)$ is constant.

Proof: Let $f(r,t) = \exp_x(rV(t))$. Then $\omega(t) = f(r(t),t)$. We have the formula

$$\frac{d\omega}{dt} = \frac{\partial f}{\partial r}\frac{dr}{dt} + \frac{\partial f}{\partial t}$$

By Gauss' lemma, the vectors $(\partial f/\partial r)(dr/dt)$ and $\partial f/\partial t$ are orthogonal. Since $\|\partial f \partial r\| = 1$, we know that

$$\left\|\frac{d\omega}{dt}\right\|^2 = \left|\frac{dr}{dt}\right|^2 + \left\|\frac{\partial f}{\partial t}\right\|^2$$

where equality holds only when $\partial f/\partial t = 0$, ie: when the $dV/dt = 0$. Hence:

$$Length(\omega) = \int_a^b \left\|\frac{d\omega}{dt}\right\| \, dt \geq \int_a^b |r'(t)| \, dt \geq |r(b) - r(a)|$$

and equality holds if and only if the function $r(t)$ is monotone and the function $V(t)$ is constant. □

Armed with the above lemma, it is now quite straightforward to prove theorem 6.33.

Proof of theorem 6.33: Consider a piecewise smooth path, ω, from the point $x \in M$ to a point $y = \exp_x(rV) \in U$, where $0 < r \leq \varepsilon$ and $\|V\| = 1$. Then for any real number $\delta \in (0, r)$ the path ω must contain a path joining the spherical sphells $S(x, \delta)$ and $S(x, r)$ and lying between these shells. By the above lemma, the length of this piece of the path ω is at least $r - \delta$.

Hence, letting δ approach zero, we see that

$$Length(\omega) \geq r = Length(\gamma)$$

If the path ω is a reparametrisation of the path γ, then the lengths of the two curves are certainly equal. If such is not the case, the conditions stated in the above lemma ensure that the above inequality is strict. □

Let us say that a path $\omega\colon [0, l] \to M$ is *parametrised by arclength* if:

$$s = \int_0^s \left\| \frac{d\omega}{dt} \right\| \, dt$$

for any point $s \in [0, l]$.

Theorem 6.33 has the following easy corollary.

Corollary 6.36 *Let $\gamma\colon [0, l] \to M$ be a path parametrised by arclength. Suppose that for every path $\omega\colon [0, l] \to M$ with $\omega(0) = \gamma(0)$ and $\omega(l) = \gamma(l)$ we have the inequality*

$$Length(\gamma) \leq Length(\omega)$$

Then the path γ is a geodesic.

Proof: The desired statement follows locally from theorem 6.33. But the condition for a path to be a geodesic *is* a purely local condition so the result follows. □

Example 6.37 Consider the space $\mathbb{R}^n$ equipped with the standard metric, and coordinates $(x^1, \ldots, x^n)$. Let $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ be a geodesic. Write $\gamma(t) = (x^1(t), \ldots, x^n(t))$.

The Christoffel symbols, Γ_{ij}^k are then all equal to zero, and the geodesic equation for the curve γ becomes:

$$\frac{d^2 x^i}{dt^2} = 0$$

Thus geodesics in $\mathbb{R}^n$ are straight lines. It follows that the shortest path between any two points of $\mathbb{R}^n$ is a straight line.

Example 6.38 Consider the circle, S^n, with the metric it inherits as the subset of $\mathbb{R}^n$ consisting of all vectors of length 1. A *great circle* in S^n is then the intersection of a two-dimensional subspace of $\mathbb{R}^n$ with S^n.

The set of geodesics in S^n is the set of segments of great circles. To see this fact, let $R\colon S^n \to S^n$ be the map defined by reflection in some two-dimensional subspace, E, of $\mathbb{R}^n$. Then R is an isometry, with fixed point set the great circle $C = S^n \cap E$.

Choose points $x, y \in C$, and let C' be the image of the unique geodesic of minimum length between them. Since the map R is an isometry, it follows that $f(C')$ is also the unique geodesic of minimal length between x and y. Thus $C' = f(C')$, and so $C' \subseteq C$.

Thus any given geodesic in S^n is a segment of a great circle. Since any two points of S^n lie on some great circle, the above argument tells us that every segment of a great circle is a geodesic.

Locally, any two points $x, y \in S^n$ that are sufficiently close together are joined by a unique geodesic of minimum length. However, by going the "wrong way" around the sphere, there is another geodesic joining them. In fact, we can define still more geodesics from x to y by winding around the sphere more than once.

If x and y are opposite points of S^n, any semicircle joining x to y is a geodesic of minimal lenght. Thus, there is no geodesics of minimal length joining the points x and y.

Definition 6.39 We call a subset $U \subseteq M$ *geodesically convex* if any two points $x, y \in U$ are joined by a unique geodesic lying entirely in U.

Theorem 6.40 *Any point $x \in M$ has a geodesically convex neighbourhood.* □

6.4 Completeness

Let M be a connected closed Riemannian manifold. We can define a metric on M (in the sense of topology, rather than in the sense of Riemannian geometry) by the formula

$$d(x, y) = \inf\{Length(\gamma) \mid \gamma \text{ is a piecewise-smooth curve from } x \text{ to } y\}$$

The results of the previous section ensure that this metric does define the usual topology on the manifold M. In this section we compare completeness of M as a metric space with the following notion.

Definition 6.41 We call M *geodesically complete* if the exponential map is always defined.

Equivalently, the manifold M is geodesically complete if a given geodesic segment $\gamma\colon I \to M$ can always be extended to a geodesic $\gamma\colon \mathbb{R} \to M$.

Theorem 6.42 (The Hopf-Rinow Theorem) *Any two points in a geodesically complete manifold can be joined by a geodesic of minimum length.*

Proof: Let M be geodesically complete. Consider two points $x, y \in M$, and let $d(x, y) = r$. We can of course assume that $r > 0$.

Choose a geodesic neighbourhood $N(x, R)$, and a positive real number $\delta < R$. Since the spherical shell $S(x, \delta)$ is compact, we can find a point $y_0 = \exp_x(\delta V) \in S(x, \delta)$ such that $d(y_0, y) \leq d(x, y)$ for all $s \in S(x, \delta)$. We claim that

$$\exp_x(rV) = y$$

Thus the geodesic segment

$$\gamma(t) = \exp_x(tV) \qquad t \in [0, r]$$

is a geodesic from x to y of minimum length. Of course, we need geodesic completeness at this final step to show that the above geodesic is defined for $r > R$.

To prove our claim, we will show that

$$d(\gamma(t), y) = r - t \qquad t \in [\delta, r]$$

Taking $t = r$ we complete the proof.

Observe that every curve from x to y must intersect the shell $S(x, \delta)$. Therefore:

$$d(x, y) = \sup\{d(x, s) + d(s, y) \mid s \in S(x, \delta)\} = \delta + d(y_0, y)$$

Thus $d(\gamma(\delta), y) = r - \delta$. Let

$$t_0 = \sup\{t \in [\delta, r] \mid d(\gamma(t), y) = r - t\}$$

By continuity, we have the formula $d(\gamma(t_0), y) = r - t_0$. Suppose $T_0 < r$. Let R' be the injectivity radius of M at $\gamma(t_0)$, and let $\delta' < \inf\{R', r - t_0\}$. Since the spherical shell $S(\gamma(t_0), \delta')$ is compact, we can find a point $y_0' \in S(\gamma(t_0), \delta')$ such that $d(y_0', y) \leq d(x, y)$ for all $s \in S'$. Observe that every path from $\gamma(t_0)$ to y must intersect the shell $S(\gamma(t_0), \delta')$. Therefore

$$d(\gamma(t_0), y) = \inf\{d(\gamma(t_0, s) + d(s, y) \mid s \in S(\gamma(t_0), \delta')\} = \delta' + d(y_0', y)$$

Thus $d(y_0', y) = (r - t_0) - \delta'$. We claim that $y_0' = \gamma(t_0 + \delta')$. By the triangle inequality for the metric d:

$$d(x, y_0') \geq d(x, y) - d(y_0', y) = t_0 + \delta'$$

But we can obtain a path of length $t_0 + \delta'$ from x to y_0' by following the geodesic γ from x to $\gamma(t_0)$, then the minimal geodesic from $\gamma(t_0)$ to y_0'. By corollary 6.36 this join is itself a geodesic, and so must coincide with the geodesic γ.

Hence $\gamma(t_0 + \delta') = y_0'$ and we have the equation

$$d(\gamma(t_0 + \delta'), y) = d(y_0', y) = r - (t_0 + \delta')$$

If $t_0 < r$ the above equation contradicts the definition

$$t_0 = \sup\{t \in [\delta, r] \mid d(\gamma(t), y) = r - t\}$$

Therefore $t_0 = r$ and we are done. □

Note that the above theorem tells us nothing about the uniqueness of minimal geodesics. For example, the sphere S^n is geodesically complete, but there is more than one geodesic of minimum length joining two opposite points of S^n.

Corollary 6.43 *A connected closed Riemannian manifold M is geodisically complete if and only if it is complete as a metric space.*

Proof: Let M be complete as a metric space. Consider a geodesic $\gamma\colon (a, b) \to M$. Let (t_n) be a sequence of points in the interval (a, b) converging to b. Then the sequence $(\gamma(t_n))$ is a Cauchy sequence in M, and so converges to a point

$x \in M$. Defining $\gamma(b) = x$ we have a geodesic $\gamma: (a, b] \to M$. By proposition 6.28 we can extend the geodesic γ past b. Similarly, we can define the geodesic γ past a.

Thus we have an extended geodesic $\gamma: (a', b') \to M$ where $a' < a$ and $b' > b$. But we can repeat the above process, so by a least upper bound argument, we obtain a geodesic $\gamma: \mathbb{R} \to M$.

Conversely, let M be geodesically complete. Let $B \subseteq M$ be a bounded subset, of diameter d. Let $x \in B$. Then by the above theorem we can find a real number $R > 0$ such that the exponential map

$$\exp_x: \overline{D}(0, R) \to M$$

maps the closed disk $\overline{D}(0, R) \subseteq T_xM$ onto a compact subset of M containing B. Hence the closure $\overline{B}$ is compact, which means that M is a complete metric space. □

We therefore do not, at least in the connected case, have to distinguish between geodesically complete Riemannian manifolds, and Riemannian manifolds that are complete as metric spaces. We refer simply to *complete Riemannian manifolds*, where we could be using either definition of completeness.

Corollary 6.44 *Let M be a connected closed compact Riemannian manifold. Then any two points in M are joined by a minimal geodesic.*

Proof: Since the manifold M is compact, it must be complete. The result now follows immediately from the above corollary and Hopf-Rinow theorem. □

6.5 Normal Bundles

Let M be a Riemannian manifold. Let N be an embedded submanifold. Then N inherits a metric from the manifold M. The tangent bundle, TN, is a subbundle of the tangent bundle TM. We define the *normal bundle*, $T^\perp N$, to be the bundle over N with fibres

$$T_x^\perp N = \{V \in T_xM \mid \langle V, W \rangle = 0 \text{ for all } W \in T_xN\}$$

Since TN is a sub-bundle of TM, we can form the quotient TM/TN, which is the bundle with fibres T_xM/T_xN. The bundles $T^\perp N$ and TM/TN have atlases of local trivialisations coming from the atlas of trivialisations for the tangent bundle TM (in the same way as the topological structure of dual bundles is defined).

The following result is straightforward.

Proposition 6.45 *The quotient bundle TM/TN and normal bundle $T^\perp N$ are smoothly isomorphic.* □

Example 6.46 Consider the sphere, S^n, embedded in $\mathbb{R}^{n+1}$ in the usual way. Then the normal bundle $T^\perp\mathbb{R}^n$ is one-dimensional. It is trivial since we can define a nowhere-vanishing section $u: S^n \to T^\perp S^n$ by the formula $u(x) = x$, where we identify the tangent space $T_x\mathbb{R}^n$ with $\mathbb{R}^n$.

Definition 6.47 Let N be an embedded submanifold of a smooth manifold M. Then a *tubular neighbourhood* of N in M is a vector bundle $p: U \to N$, where U is an open neighbourhood of N in M.

Before we prove the existence of tubular neighbourhoods, we need a topological lemma.

Lemma 6.48 *Let X be a compact metric space, and let $X_0 \subseteq X$ be a closed subset. Let $f: X \to Y$ be a local homeomorphism such that $f|_{X_0}$ is injective. Then there is a real number $\varepsilon > 0$ such that the map f is injective when restricted to the open neighbourhood*

$$N(X_0, \varepsilon) = \{x \in X \mid d(x, X_0) < \varepsilon\}$$

Proof: Let

$$C = \{(x_0, y_0) \in X \times X \mid x \neq y, f(x) = f(y)\}$$

The set C is closed because the map f is a local homeomorphism. Consider the map $g: C \to \mathbb{R}$ defined by the formula

$$g(x, y) = d(x, X_0) + d(y, X_0)$$

Since the set C, as a closed subset of a compact metric space, is compact, we can find a real numbet $\varepsilon > 0$ such that $g(x, y) \geq 2\varepsilon$ for all $(x, y) \in C$. It follows that the map f is injective on the ε-neighbourhood of X_0. □

Theorem 6.49 (The Tubular Neighbourhood Theorem) *Let N be a compact embedded submanifold of a manifold M. Then there is a tubular neighbourhood of N in M which is equivalent to the normal bundle of N in M.*

Proof: By theorem 5.6 we can choose a Riemannian metric $\langle -, - \rangle$ for M, with corresponding norm $\| - \|$ and metric d. For $\varepsilon > 0$, let us write

$$E_\varepsilon = \{V \in T_x^\perp N \mid x \in N, \|V\| < \varepsilon\}$$

and

$$U_\varepsilon = \{x \in M \mid d(x, N) < \varepsilon\}$$

By theorem 6.31 and compactness of the manifold N the exponential map $(x, V) \mapsto \exp(x, V)$ is defined on E_ε for some $\varepsilon > 0$. We claim that the exponential map is a diffeomorphism provided ε is sufficiently small. The theorem clearly follows from this claim.

Let $V \subseteq E_\varepsilon$ be the set of regular points for the smooth map exp. Let us consider N to be the set of zero tangent vectors in the space E_ε. Then certainly $V \supseteq N$, and $\overline{V_1} = \overline{V \cap E_1}$ is compact. Since the exponential function is injective on $N \subseteq V_1$, by the above lemma the exponential function is injective on E_ε provided ε is sufficiently small.

Certainly $\exp[E_\varepsilon] \subseteq U_\varepsilon$. We need to prove that $\exp[E_\varepsilon] = U_\varepsilon$. Choose $y \in U_\varepsilon$. Since N is compact, there is a point $x_0 \in N$ such that $d(x_0, y) \leq d(x, y)$ for all $x \in N$. Let $\gamma: [0, 1] \to N$ be the unique geodesic of length less that ε such that $\gamma(0) = x_0$ and $\gamma(1) = y$.

The fact that x_0 is the nearest point on N to the point y means that the tangent vector $\gamma'(0) \in T_{x_0}M$ is normal to N. Hence

$$y = \exp(x_0, \gamma'(0)) \qquad \gamma'(0) \in E_\varepsilon$$

and we are done. □

In fact the tubular neighbourhood theorem is also true for non-compact submanifolds. However, we will not give the proof here.

It is interesting to see that some fairly deep facts in Riemannian geometry are used to determine the existence of tubular neighbourhoods— a statement which does not itself even mention Riemannian geometry. The same is true of the following result, which has a similar proof.

Theorem 6.50 (Collared Neighbourhood Theorem) *Let M be a manifold with compact boundary ∂M. Then ∂M has a neighbourhood in M that is diffeomorphic to $\partial M \times [0,1)$.* □

7 Some Notions from Algebraic Topology

7.1 Homotopy

When looking at topological spaces, it is often easy to see when two spaces are homeomorphic; we simply construct a homeomorphism! For example the open interval $(-1,1)$ and the entire real line $\mathbb{R}$ are homeomorphic; we can define a homeomorphism $f\colon(-1,1) \to \mathbb{R}$ by the formula

$$f(x) = \frac{x}{1-|x|}$$

In general a far more difficult problem is to prove that two spaces are *not* homeomorphic. A *homeomorphism-invariant* is a property of a space that is preserved by the relation of homeomorphism. For example, compactness and connectedness are homeomorphism-invariants. Hence the intervals $[0,1]$ and $(0,1)$ are not homeomorphic because the space $[0,1]$ is compact whereas the space $(0,1)$ is not. The spaces $[0,3]$ and $[0,1]\cup[2,3]$ are not homeomorphic because the space $[0,3]$ is connected whereas the space $[0,1]\cup[2,3]$ is not.

The goal of algebraic topology is to associate invariants to topological spaces that enable us to tell whether or not they are homeomorphic.

Generally the notion of homeomorphism is rather restrictive; in algebraic topology it is common to work with a more general equivalence relation.

Definition 7.1 Let $f,g\colon X \to Y$ be continuous maps. Then we say the maps f and g are *homotopic* if there is a continuous map $F\colon X\times[0,1] \to Y$ such that $F(-,0) = f$ and $F(-,1) = g$.

The map F in the above definition is sometimes termed a *homotopy* between the map f and the map g.

Proposition 7.2 *The notion of homotopy is an equivalence relation on the set of all continuous maps from the space X to the space Y.*

Proof: It is obvious that the relation of homotopy is reflexive and symmetric. We prove transitivity.

Let F be a homotopy between a map $f: X \to Y$ and a map $g: X \to Y$. Let G be a homotopy between the map $g: X \to Y$ and a map $h: X \to Y$. Then we can define a homotopy $F \star G: X \times [0,1] \to Y$ between the maps f and h by the formula

$$F \star G(x,t) = \begin{cases} F(x,2t) & t \leq \frac{1}{2} \\ G(x,2t-1) & t \geq \frac{1}{2} \end{cases}$$

□

The set of all maps $g: X \to Y$ that are homotopic to a map $f: X \to Y$ is thus called the *homotopy class* of the map f.

Definition 7.3 A continuous map $f: X \to Y$ is said to be a *homotopy equivalence* if there is a continuous map $g: Y \to X$ such that the composites gf and fg are homotopic to the identity maps 1_X and 1_Y respectively.

Spaces X and Y are said to be *homotopy-equivalent* if there exists a homotopy equivalence $f: X \to Y$. It is easy to see that homeomorphic spaces must be homotopy equivalent.

Example 7.4 Euclidean space $\mathbb{R}^n$ is homotopy-equivalent to a single point $\{0\}$. To see this fact, define maps $f: \mathbb{R}^n \to \{0\}$ and $g: \{0\} \to \mathbb{R}^n$ by the formulae $f(x) = 0$ and $g(0) = 0$ respectively. The composite $f \circ g$ is the identity map $1_{\{0\}}$. We can define a homotopy between the composite $g \circ f$ and the identity map $1_{\mathbb{R}^n}$ by the formula

$$F(v, tv) = tv; \qquad t \in [0,1], v \in \mathbb{R}^n$$

A space that is homotopy-equivalent to a single point is called *contractible*. A space is contractible if and only if there is a homotopy between the constant map onto a single point in that space and the identity map.

Definition 7.5 We define $\pi_0(X)$ to be the set of path-components of a topological space.

Let us write $\langle x \rangle$ to denote the path-component of a point $x \in X$. Then a continuous map $f: X \to Y$ induces a map $f_\star: \pi_0(X) \to \pi_0(Y)$ by the formula

$$f_\star \langle x \rangle = \langle f(x) \rangle$$

If maps $f, g: X \to Y$ are homotopic, the induced maps $f_\star, g_\star: \pi_0(X) \to \pi_0(Y)$ are equal. It follows that if a map $f: X \to Y$ is a homotopy-equivalence, the induced map $f_\star: \pi_0(X) \to \pi_0(Y)$ is a bijection.

Definition 7.6 Let $f, g: M \to N$ be smooth maps. Then we say the maps f and g are *smoothly homotopic* if there is a smooth map $F: X \times [0,1] \to Y$ such that $F(-,0) = f$ and $F(-,1) = g$.

As in definition 7.3 we can talk about a smooth map $f: M \to N$ being a *smooth homotopy equivalence*.

Theorem 7.7 *Let $f: M \to N$ be a continuous map betweem smooth compact manifolds. Then f is homotopic to a smooth map.*

Proof: In this proof, we need a small amount of functional analysis. Let X be a compact Hausdorff topological space, and let $C(X)$ be the set of continuous functions $f: X \to \mathbb{R}$. The set $C(X)$ is a normed vector space; the operations of addition and scalar multiplication are defined pointwise, and the norm is defined by the formula

$$\|f\| = \sup\{|f(x)| \mid x \in X\}$$

Let us call a subset $A \subseteq C(X)$ a *subalgebra* if for any two functions $f, g \in A$ the product, fg, defined pointwise, also belongs to the set A. We say that the algebra A *separates points* of X if for any two points $x, y \in X$ with $x \neq y$ we can find a function $f \in A$ such that $f(x) = 0$ but $f(y) \neq 0$.

Theorem [The Stone-Weierstrass Theorem] Let A be a subalgebra of $C(X)$ that contains the constant functions and separates points. Then A is a dense subset of $C(X)$. □

Note that the Stone-Weierstrass theorem implies that the set of smooth functions of M, $C^\infty(M)$, is a dense subset of the normed space $C(M)$. It is this fact that we need here.

Suppose $N \subseteq \mathbb{R}^N$. Write

$$f(x) = (f_1(x), \ldots, f_N(x)) \qquad x \in M$$

Then for any real number $\varepsilon > 0$ we can find smooth functions $g_i: M \to \mathbb{R}$ such that $|f_i(x) - g_i(x)| < \varepsilon$ for all $x \in M$. Hence, by the tubular neighbourhood theorem, there is a smooth map $g: M \to \mathbb{R}^N$ such that that the image $g[M]$ lies in some tubular neighbourhood, U, of N in $\mathbb{R}^N$.

According to example **??**, the structure map $p: U \to N$ is a smooth homotopy-equivalence. We therefore obtain a smooth map

$$p \circ g: M \to N$$

that is homotopic to f. □

The above argument is the one used to fill in the details in the proof of the Brouwer fixed point theorem when we passed from consideration of smooth maps to continuous maps.

Corollary 7.8 *Let $f, g: M \to N$ be smooth maps between compact manifolds. Suppose that the maps f and g are homotopic. Then they are smoothly homotopic.* □

7.2 Categories and Functors

Algebraic topology is frequently best expressed within an abstract language called *category theory*.

Definition 7.9 A *category*, $\mathcal{C}$, consists of a collection $Ob(\mathcal{C})$ of *objects* and a set $Hom(A, B)$ of *morphisms* for any two objects $A, B \in Ob(\mathcal{C})$ such that the following axioms are satisfied:

- For all morphisms $f \in Hom(A, B)$ and $g \in Hom(B, C)$ there is assigned a morphism $gf \in Hom(A, C)$, called the *composition* of f and g.
- Composition is *associative*, that is to say the formula $(hg)f = h(gf)$ holds for all morphisms $f \in Hom(A, B)$, $g \in Hom(B, C)$, and $h \in Hom(C, D)$.
- For all objects $A \in Ob(\mathcal{A})$ there is a morphism $1_A \in Hom(A, A)$, called the *identity at* A, such that $1_A g = g$ and $f 1_A = f$ for all morphisms $f \in Hom(A, B)$ and $g \in Hom(B, A)$.

For instance we may form the category of sets. The collection of objects is the collection of all sets. For sets A and B the morphism set $Hom(A, B)$ consists of all maps from A to B. This example indicates that the collection of objects in a given category is not in general a set.

Other examples include the category of all abelian groups and group homomorphisms, the category of all real vector spaces and linear maps, and the category of all topological spaces and continuous maps. Indeed we generally think of a category as a collection of certain algebraic objects and a morphism set as a collection of homomorphisms between these algebraic objects. Because of this idea we sometimes use the notation $f: A \to B$ to mean that f is an element of the morphism set $Hom(A, B)$. We call a morphism $f: A \to B$ an *invertible morphism* or an *isomorphism* if there is a morphism $g: B \to A$ such that $gf = 1_A$ and $fg = 1_B$.

Two more examples that are relevant in these notes are the category of smooth manifolds and smooth maps, and the category of vector bundles and bundle maps over a given topological space.

The morphisms of a category need not be actual maps. We can see this by mentioning another example of importance to us in these notes: the category of topological spaces and homotopy classes of continuous maps. We call this category the *homotopy category*.

The objects of a given category are not always sets. To see this, let P be a partially ordered set. Then P is a category; the objects are the elements of the set P and there is precisely one morphism in the set $Hom(i, j)$ whenever $i \leq j$. To take another example, let G be a group. Then G is a category with just one object; the morphisms from this object to itself are simply the elements of G.

Definition 7.10 Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then a *functor* $F: \mathcal{C} \to \mathcal{D}$ is a procedure that assigns an object $F(A) \in Ob(\mathcal{D})$ to each object $A \in Ob(\mathcal{C})$ and a morphism $f_\star \in Hom(F(A), F(B))$ to each morphism $f \in Hom(A, B)$. The induced morphisms $f_\star$ must satisfy the formulae:

$$(fg)_\star = f_\star g_\star \qquad (1_A)_\star = 1_{F(A)}$$

Actually, a functor as defined above is usually called a *covariant functor*. There is a dual notion of a *contravariant functor*; a contravariant functor $F: \mathcal{C} \to \mathcal{D}$ is a procedure that assigns an object $F(A) \in Ob(\mathcal{D})$ to each object $A \in Ob(\mathcal{C})$ and a morphism $f_\star \in Hom(F(B), F(A))$ to each morphism $f \in Hom(A, B)$ such that $(fg)_\star = g_\star f_\star$ and $(1_A)_\star = 1_F(A)$.

A trivial example of a (covariant) functor is the *identity functor* $1: \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$, which takes any object or morphism to itself.

Example 7.11 Let V be a real vector space. Then the assignment $V \mapsto V^\star$ taking V to its dual space is a contravariant functor from the category of real vector spaces to itself. Given a linear map $T: V \to W$ we define the induced map $T^\star: W^\star \to V^\star$ by the formula

$$T^\star(\hat{w})(v) = \hat{w}(Tv)$$

Example 7.12 The assignment $X \mapsto \pi_0(X)$, taking a topological space X to its set of path components is a covariant functor from the category of topological spaces to the category of sets.

If we write $\langle x \rangle$ to denote the path-component of a point $x \in X$. Then a continuous map $f: X \to Y$ induces a map $f_\star: \pi_0(X) \to \pi_0(Y)$ by the formula

$$f_\star \langle x \rangle = \langle f(x) \rangle$$

The above example can also be considered a functor from the homotopy category to the category of sets. Generalising, we can state a more formal definition of what we mean by a *homotopy-invariant*; it is any functor from the homotopy category to some other category.

Definition 7.13 Let $F, G: \mathcal{C} \to \mathcal{D}$ be functors. Then a *natural transformation* $g: F \to G$ consists of a morphism $g_A \in Hom(F(A), G(A))$ for each object $A \in Ob(\mathcal{C})$ such that $G(f)g_A = g_B F(f)$ for any morphism $f \in Hom(A, B)$.

We call a natural transformation $g: F \to G$ a *natural isomorphism* if the morphisms $g_A \in Hom(F(A), G(A))$ are isomorphisms for all A.

We can express the condition $G(f)g_A = g_B F(f)$ by saying that we have a *commutative diagram*

$$\begin{array}{ccc} F(A) & \xrightarrow{g_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ F(B) & \xrightarrow{g_B} & G(B) \end{array}$$

More complicated commutative diagrams are fairly common in category theory and algebraic topology, and have the obvious meaning. Proofs involving commutative diagrams are sometimes called *diagram chases*.

Example 7.14 Let $\mathcal{V}^2$ be the category in which the objects are pairs, (V_1, V_2) of vector spaces, and the morphisms are pairs (α_1, α_2) of linear transformations. Define functors T_1 and T_2 from the category $\mathcal{V}^2$ to the category of vector spaces and linear transformations by the formulae

$$T_1(V_1, V_2) = V_1 \otimes V_2 \qquad T_1(\alpha_1, \alpha_2) = \alpha_1 \otimes \alpha_2$$

and

$$T_2(V_1, V_2) = V_2 \otimes V_1 \qquad T_2(\alpha_1, \alpha_2) = \alpha_2 \otimes \alpha_1$$

Then we can define a natural isomorphism $g: T_1 \to T_2$ by the formula

$$g_{(V_1, V_2)}(u_1 \otimes u_2) = u_2 \otimes u_1$$

Example 7.15 Let V be a vector space. Then we have an isomorphism $\alpha_V: V \to (V^\star)^\star$ from the space V to its second dual space defined by the formula

$$\alpha_V(v)(f) = f(v)$$

for all vectors $v \in V$ and functionals $f \in V^\star$. For any linear map $T: V \to W$ we know that $\alpha_W T = (T^\star)^\star \alpha_V$ so we have a commutative diagram

$$\begin{array}{ccc} V & \stackrel{\alpha_V}{\to} & (V^\star)^\star \\ \downarrow & & \downarrow \\ W & \stackrel{\alpha_W}{\to} & (W^\star)^\star \end{array}$$

Thus the collection of maps $\alpha: V \to (V^\star)^\star$ is a natural transformation between the identity functor on the category of vector spaces and the functor taking a vector space to its second dual space.

On the category of finite-dimensional vector spaces, the natural transformation α is a natural isomorphism.

7.3 Some Homological Algebra

Definition 7.16 A *cochain complex*, $A^\bullet$, is a sequence of abelian groups and homomorphisms

$$\cdots \to A^{p-1} \stackrel{d_{p-1}}{\to} A^p \stackrel{d_p}{\to} A^{p+1} \to \cdots$$

such that $d_p d_{p-1} = 0$ for all integers $k \in \mathbb{Z}$

The homomorphisms d_p are called *differentials*. We frequently simplify notation and write d for each differential. The defining condition for a sequence to be a cochain complex is then written $d^2 = 0$.

Example 7.17 The de Rham complex of a smooth manifold is a cochain complex.

Notice that for any cochain complex we have the inclusion $im(d: A^{p-1} \to A^p) \subseteq \ker(d: A^p \to A^{p+1})$. The cochain complex $A^\bullet$ is called *exact at p* if $im(d: A^{p-1} \to A^p) = \ker(d: A^p \to A^{p+1})$. The cochain complex $A^\bullet$ is called *exact* if it is exact at all integers $p \in \mathbb{Z}$. We sometimes refer to an exact cochain complex as an *exact sequence*.

Definition 7.18 A *short exact sequence* is an exact complex of the form

$$0 \to A \stackrel{\alpha}{\to} B \stackrel{\beta}{\to} C \to 0$$

The above short exact sequence

$$0 \to A \stackrel{\alpha}{\to} B \stackrel{\beta}{\to} C \to 0$$

is called *split exact* if the group B can be written as a direct sum $\alpha[A] \oplus C'$ and the restriction $\beta|^{C'}: C' \to C$ is an isomorphism.

The proof of the following is a 'diagram chase' and is more interesting to work out yourself than read.

Proposition 7.19 5-*lemma*
Suppose we have a commutative diagram

$$\begin{array}{ccccccccc} A & \to & B & \to & C & \to & D & \to & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \to & B' & \to & C' & \to & D' & \to & E' \end{array}$$

in which the rows are exact sequences and all of the vertical maps except for the central one are isomorphisms. Then the central vertical map is also an isomorphism. □

The proof of the following result is another diagram chase.

Proposition 7.20 *Suppose we have a commutative diagram*

$$\begin{array}{ccccccccccc} \to & A^p & \xrightarrow{i} & B^p & \xrightarrow{j} & C^p & \xrightarrow{k} & A^{p+1} & \to \\ & \alpha\downarrow & & \beta\downarrow & & \downarrow\gamma & & \downarrow\alpha & \\ \to & A'^p & \xrightarrow{i'} & B'^p & \xrightarrow{j'} & C'^p & \xrightarrow{k'} & A'^{p+1} & \to \end{array}$$

in which the rows are exact and the vertical maps are all isomorphisms. Then there is an exact sequence

$$\longrightarrow A^p \overset{(\alpha,-i)}{\longrightarrow} A'^p \oplus B^p \overset{i'+\beta}{\longrightarrow} B'^p \overset{k\gamma^{-1}j'}{\longrightarrow} A^{p-1} \longrightarrow$$

□

The notion of the *cohomology* of a cochain complex is a measure of how far it is from being exact.

Definition 7.21 Let $A^\bullet$ be a cochain complex. Then we define the *cohomology groups* by the formula

$$H^p(A^\bullet) = \frac{\ker(d\colon A^p \to A^{p+1})}{im(d\colon A^{p-1} \to A^p)}$$

Observe that a cochain complex is exact if and only if its cohomology groups are all trivial.

Definition 7.22 A *morphism* of cochain complexes, $\alpha\colon A^\bullet \to B^\bullet$, is a collection of homomorphisms $\alpha\colon A^p \to B^p$ such that $\alpha d = d\alpha$.

We can form the category of all cochain complexes and morphisms. Similarly we can form the category of all short exact sequences and morphisms. A morphism of cochain complexes is sometimes called a *chain map*.

Example 7.23 A smooth map $f\colon M \to N$ induces a chain map $f^\star\colon \Omega^\bullet(N) \to \Omega^\bullet(M)$ between the associated de Rham complexes.

Proposition 7.24 *The cohomology groups are covariant functors from the category of cochain complexes to the category of abelian groups.*

Proof: Let $\alpha: A^\bullet \to B^\bullet$ be a morphism of cochain complexes. Then for any element $x \in A^p$:

$$\begin{array}{ccccc} dx = 0 & \Rightarrow & \alpha dx = 0 & \Rightarrow & d(\alpha x) = 0 \\ x = dy & \Rightarrow & \alpha x = \alpha(dy) & \Rightarrow & \alpha x = d(\alpha y) \end{array}$$

Thus $\alpha[\ker(d: A^p \to A^{p+1})] \subseteq \ker(d: B^p \to B^{p+1})$ and $\alpha[im(d: A^{p-1} \to A^p)] \subseteq im(d: B^{p-1} \to B^p)$ and we have a well-defined induced map

$$\alpha_\star: H^p(A^\bullet) \to H^p(B^\bullet)$$

It is easy to see that with such induced maps the assignment $A^\bullet \mapsto H^p(A^\bullet)$ is a covariant functor. □

We can similarly define morphisms of chain complexes. The analogue of the above result holds for homology theories.

Definition 7.25 Let $\alpha, \beta: A^\bullet \to B^\bullet$ be morphisms of cochain complexes. Then a *chain homotopy* in between α and β is a collection of homomorphisms $T: A^p \to B^{p+1}$ such that:

$$\alpha - \beta = Td \pm dT$$

We call morphisms α and β *chain homotopic* if there exists a chain homotopy between them.

Proposition 7.26 *Let $\alpha, \beta: A^\bullet \to B^\bullet$ be chain homotopic morphisms of cochain complexes. Then the induced maps $\alpha_\star, \beta_\star: H^p(A^\bullet) \to H^p(B^\bullet)$ are equal.*

Proof: Let T be a chain homotopy between the morphisms α and β. Let $x \in A^p$ and suppose that $dx = 0$. Then:

$$\alpha(x) - \beta(x) = T(dx) \pm dT(x) = d(\pm T(x)) \in im(d: B^{p+1} \to B^p)$$

Hence $\alpha^\star([x]) - \beta^\star([x]) = 0$ for all elements $[x] \in H^p(A^\bullet)$. □

We call a morphism $\alpha: A^\bullet \to B^\bullet$ a *chain homotopy equivalence* if there is a morphism $\beta: B^\bullet \to A^\bullet$ such that the compositions $\alpha\beta$ and $\beta\alpha$ are chain homotopic to the idendities $1_{B^\bullet}$ and $1_{A^\bullet}$ respectively. By the above result a chain homotopy equivalence of cochain complexes induces an isomorphism of cohomology groups.

A cochain complex $A^\bullet$ is called *chain contractible* if the identity and zero morphisms are chain homotopic. The above result tells us that the homology groups of a chain contractible cochain complex are all trivial.

Definition 7.27 A *short exact sequence* of cochain complexes is a sequence of the form

$$0 \to A^\bullet \stackrel{\alpha}{\to} B^\bullet \stackrel{\beta}{\to} C^\bullet \to 0$$

such that each sequence of abelian groups

$$0 \to A^p \stackrel{\alpha}{\to} B^p \stackrel{\beta}{\to} C^p \to 0$$

is a short exact sequence.

We can form the category of all short exact sequences of cochain complexes; the morphisms are commutative diagrams

$$\begin{array}{ccccccccc} 0 & \to & A^\bullet & \to & B^\bullet & \to & C^\bullet & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A'^\bullet & \to & B'^\bullet & \to & C'^\bullet & \to & 0 \end{array}$$

in which the vertical maps are morphisms of cochain complexes.

Theorem 7.28 *Snake Lemma*
Let

$$0 \to A^\bullet \stackrel{\alpha}{\to} B^\bullet \stackrel{\beta}{\to} C^\bullet \to 0$$

be a short exact sequence of cochain complexes. Then there are natural maps $\partial\colon H^p(C^\bullet) \to H^{p+1}(A^\bullet)$ *such that we have a long exact sequence*

$$\cdots \to H^p(A^\bullet) \stackrel{\alpha^\star}{\to} H^p(B^\bullet) \stackrel{\beta^\star}{\to} H^p(C^\bullet) \stackrel{\partial}{\to} H^{p+1}(A^\bullet) \to \cdots$$

Proof: We have a commutative diagram

$$\begin{array}{ccccccccc} & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A^{p-1} & \stackrel{\alpha}{\to} & B^{p-1} & \stackrel{\beta}{\to} & C^{p-1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A^p & \stackrel{\alpha}{\to} & B^p & \stackrel{\beta}{\to} & C^p & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A^{p+1} & \stackrel{\alpha}{\to} & B^{p+1} & \stackrel{\beta}{\to} & C^{p+1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \end{array}$$

in which the rows are exact and the vertical maps are the differentials, d.

Let $x \in C^p$ and suppose that $dx = 0$. Then $x = \beta y$ and $d\beta y = 0$ so $\beta dy = 0$. Hence $dy = \alpha z$ for some element $z \in A^{p+1}$. The claim is that we can define a natural homorphism $\partial\colon H^p(C^p) \to H^p(A^{p+1})$ by writing $\partial([x]) = [z]$ where

$$x = \beta y \qquad dy = \alpha z$$

The above argument shows that a suitable element $z \in A^{p+1}$ exists. Further, $\alpha dz = d\alpha z = d^2 y = 0$ so $dz = 0$ since the homomorphism α is injective. If we can also write $x = \beta y'$ and $dy' = \alpha z'$ then:

$$\begin{array}{rcl} \beta(y - y') = 0 & \Rightarrow & y - y' = \alpha w \\ & \Rightarrow & dy - dy' = d\alpha w = \alpha dw \\ & \Rightarrow & \alpha(z - z') = \alpha dw \\ & \Rightarrow & z - z' = dw \end{array}$$

so the elements $[z], [z'] \in H^{p+1}(A^\bullet)$ are equal. Hence the homomorphism ∂ is well-defined.

It is left as an exercise to show that the homomorphism ∂ is natural and that the sequence of homology groups

$$\cdots \to H^p(A^\bullet) \stackrel{\alpha^\star}{\to} H^p(B^\bullet) \stackrel{\beta^\star}{\to} H^p(C^\bullet) \stackrel{\partial}{\to} H^{p+1}(A^\bullet) \to \cdots$$

is exact. The proof consists of a number of 'diagram chases' similar to the above. □

The notion of a split exact sequence can sometimes be useful in homological computations.

Proposition 7.29 *A split exact sequence*

$$0 \to A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C^\bullet \to 0$$

of cochain complexes induces short exact sequences

$$0 \to H^p(A^\bullet) \xrightarrow{\alpha^\star} H^p(B^\bullet) \xrightarrow{\beta^\star} H^p(C^\bullet) \to 0$$

of cohomology groups. □

8 De Rham Cohomology

8.1 Differential Forms

Let M be a smooth manifold. Consider the space $\Gamma^0_p(M)$ of tensor fields of type $(0, p)$. We have a subspace, N, generated by tensor fields of the form

$$\alpha^1 \otimes \cdots \otimes \alpha^p + \alpha^{\sigma(1)} \otimes \cdots \otimes \alpha^{\sigma(p)}$$

where α_i is a covector field and σ is an odd permutation of the set $\{1, \ldots, p\}$.

Definition 8.1 We define the set of *differential forms* of *order p* to be the quotient

$$\Omega^p(M) = \Gamma^0_p(M)/N$$

A differential form of order p is sometimes referred to as a *p-form*.

Observe that a differential form of order 0 is just a function $f \in C^\infty(M)$. A differential form of order 1 is the same thing as a covector field.

We define the *support* of a p-form ω to be the closure of the set of points $x \in M$ such that $\omega(x) \neq 0$. In parts of the theory it is important to consider compactly supported differential forms. We write $\Omega^p_c(M)$ to denote the set of differential forms of order p with compact support.

Of course, on a compact manifold every differential form has compact support.

Definition 8.2 We define the *exterior product*

$$\wedge \colon \Omega^p(M) \times \Omega^q(M) \to \Omega^{p+q}(M)$$

by the formula

$$[\alpha] \wedge [\beta] = [\alpha \otimes \beta]$$

where $\alpha \in \Gamma^0_p(M)$ and $\beta \in \Gamma^0_p(M)$.

Note that the product of a differential form with a compactly supported differential form is compactly supported.

Now, observe that any differential p-form can be written as a sum of p-forms of the type

$$\alpha^1 \wedge \cdots \wedge \alpha^p$$

where α^i is a covector field. The definition of the space $\Omega^p(M)$ means that we have the identity

$$\alpha^1 \wedge \cdots \wedge \alpha^p = sgn(\sigma)\alpha^{\sigma(1)} \wedge \cdots \wedge \alpha^{\sigma(p)}$$

whenever σ is a permutation of the set $\{1, \ldots, p\}$. The following result is easily deduced.

Proposition 8.3 *Consider differential forms $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(M)$. Then we have the identity*

$$\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega$$

□

Corollary 8.4 *Consider a differential form $\omega \in \Omega^p(M)$ where p is odd. Then:*

$$\omega \wedge \omega = 0$$

□

Proposition 8.5 *Let $\omega \in \Omega^p(M)$. Then in coordinates $(x^1, \ldots, x^n)$ defined by some chart we can write*

$$\omega = \sum_{i_1 < \cdots < i_p} f_{i_1, \ldots, i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

where $f_{i_1, \ldots, i_p}$ is a smooth function.

Proof: Let us write

$$\omega = \sum_j \alpha^{j_1} \wedge \cdots \wedge \alpha^{j_p}$$

where α^{j_k} is a covector field. In coordinates $(x^1, \ldots, x^n)$ defined by a chart we can certainly write

$$\alpha^{j_k} = g_1^{j_k} dx^1 + \cdots + g_n^{j_k} dx^n$$

where g^{j_k} is a smooth function.

Hence

$$\omega = \sum_{i,j} (g_1^{j_1} dx^1 + \cdots + g_n^{j_1} dx^n) \wedge \cdots \wedge (g_1^{j_p} dx^1 + \cdots + g_n^{j_p} dx^n)$$

Bilinearity of the exterior product enables us to write:

$$\omega = \sum_{j_1, \ldots, j_p} h_{j_1, \ldots, j_p} dx^{j_1} \wedge \cdots \wedge dx^{j_p}$$

But the expression $dx^{j_1} \wedge \cdots \wedge dx^{j_p}$ is either zero (if $dx^{j_k} = dx^{j_l}$ for some $k \neq l$) or else takes the form

$$\pm dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

where $i_1 < \cdots < i_p$. Hence:

$$\omega = \sum_{i_1 < \cdots < i_p} f_{i_1,\ldots,i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

as required. □

A slightly more condensed notation can be convenient when dealing with differential forms and coordinates. Let us write I to denote a p-tuple $(i_1, \ldots, i_p)$, and dx^I to denote the product $dx^{i_1} \wedge \cdots \wedge dx^{i_p}$. Then for coordinates defined by some chart a differential form $\omega \in \Omega^p(M)$ can be written

$$\omega = \sum_I f_I dx^I$$

Corollary 8.6 *Let M be a smooth manifold of dimension n. then $\Omega^p(M) = 0$ if $p > n$.*

Proof: In coordinates $(x^1, \ldots, x^n)$ there are no expressions of the form

$$dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

where $i_1 < \cdots < i_p$. □

8.2 The de Rham Complex

Recall that for a smooth function $f \in C^\infty(M)$ we can define a covector field by the formula

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i$$

in coordinates defined by a chart. We can extend this definition.

Definition 8.7 Consider a differential form $\omega \in \Omega^p(M)$. Write in coordinates defined by a chart

$$\omega = \sum_I f_I dx^I$$

Then we define a $(p+1)$-form $d\omega \in \Omega^{p+1}(M)$ by the formula

$$d\omega = \sum_I df_I \wedge dx^I$$

The operator $d\colon \Omega^p(M) \to \Omega^{p+1}(M)$ is called the (de Rham) *differential.* One way to prove that the differential is well-defined is by brute force and lemma 9.15— a calculation with the chain rule shows that the form $d\omega$ defined as above is independent of any choice of chart. Here we describe a more elegant method.

Proposition 8.8 *Let (U, V, ϕ) be a chart. Then the differential $d: \Omega^p(U) \to \Omega^{p+1}(U)$ defined by the above formula has the following properties:*

- $d(\omega + \eta) = d\omega + d\eta$ *for all* $\omega, \eta \in \Omega^p(U)$.
- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$ *for all* $\omega \in \Omega^p(U)$ *and* $\eta \in \Omega^q(U)$.
- $d(d\omega) = 0$ *for all* $\omega \in \Omega^p(U)$.

Proof: The first of these formulae is obvious. To prove the second, observe that the first formula means we can assume that

$$\omega = f dx^I \qquad \eta = g dx^J$$

where f and g are smooth function. In this case:

$$\begin{aligned} d(\omega \wedge \eta) &= d(fg) \wedge dx^I \wedge dx^J \\ &= gdf \wedge dx^I \wedge dx^J + fdg \wedge dx^I \wedge dx^J \\ &= d\omega \wedge \eta + (-1)^p f dx^I \wedge (dg \wedge dx^J) \\ &= d\omega \wedge \eta + \omega \wedge d\eta \end{aligned}$$

To prove the last formula, we again consider the case $\omega = f dx^I$ so that

$$d\omega = \sum_i \frac{\partial f}{\partial x^i} dx^i \wedge dx^I$$

and

$$d(d\omega) = \sum_{i,j} \frac{\partial^2 f}{\partial x^j x^i} dx^j \wedge dx^i \wedge dx^I = \sum_{i<j} \frac{\partial^2 f}{\partial x^j x^i} (dx^j \wedge dx^i + dx^i \wedge dx^j) \wedge dx^I = 0$$

since $dx^i \wedge dx^i = 0$ and $dx^i \wedge dx^j = -dx^j \wedge dx^i$ □

The last of the above formulae is frequently abbrebriated by writing '$d^2 = 0$'.

Proposition 8.9 *Let (U, V, ϕ) be a coordinate chart on M with local coordinates $(x^1, \ldots, x^n)$. Suppose we have an operator $d': \Omega^p(U) \to \Omega^{p+1}(U)$ with the following properties.*

- $d'(\omega + \eta) = d'\omega + d'\eta$ *for all* $\omega, \eta \in \Omega^p(U)$.
- $d'(\omega \wedge \eta) = d'\omega \wedge \eta + (-1)^p \omega \wedge d'\eta$ *for all* $\omega \in \Omega^p(U)$ *and* $\eta \in \Omega^q(U)$.
- $d'(d'\omega) = 0$ *for all* $\omega \in \Omega^p(U)$.
- $d'f = df$ *when f is a smooth function.*

Then $d' = d$.

Proof: The first property means that it suffices to show that $d'(f dx^I) = d(f dx^I)$ when f is a smooth function on U. The second and last properties yield the formula

$$d'(f x^I) = d'f \wedge dx^I + f d'(dx^I) = df \wedge dx^I + f \wedge d'(dx^I)$$

With the first property, we can write

$$dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k} = d'x^{i_1} \wedge \cdots \wedge d'x^{i_k}$$

We know that $d'(d'x^{i_l}) = 0$ for each i_l. It follows by induction using the third formula that $d'(dx^I) = 0$ and we are done. □

Corollary 8.10 *Let M be a smooth manifold. Then there is a unique operator $d\colon \Omega^p(M) \to \Omega^{p+1}(M)$ such that:*

- $d(\omega + \eta) = d\omega + d\eta$ *for all* $\omega, \eta \in \Omega^p(U)$.
- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$ *for all* $\omega \in \Omega^p(U)$ *and* $\eta \in \Omega^q(U)$.
- $d(d\omega) = 0$ *for all* $\omega \in \Omega^p(U)$.
- *Let* $f \in C^\infty(M)$. *Then*
$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i$$
in terms of local coordinates $x^1, \ldots, x^n$.

Proof: For each chart (U, V, phi) we have a unique operator $d_U\colon \Omega^p(U) \to \Omega^{p+1}(U)$ satisfying the above properties. We can therefore define a suitable operator $d\colon \Omega^p(M) \to \Omega^{p+1}(M)$ by writing

$$d\omega(x) = d_U(\omega|_U)(x)$$

if $x \in U$. □

Because of proposition 8.9 the operator d is defined in terms of local coordinates by our original formula

$$d\omega = \sum_I df_I \wedge dx^I$$

where

$$\omega = \sum_I f_I dx^I$$

Definition 8.11 The sequence of groups and homomorphisms

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots$$

is called the *de Rham complex* of the manifold M.

8.3 De Rham Cohomology

Definition 8.12 The cohomology groups of the de Rham complex:

$$H^p(M) = \frac{\ker(d\colon \Omega^p(M) \to \Omega^{p+1}(M))}{im(d\colon \Omega^{p-1}(M) \to \Omega^p(M))}$$

are called the *de Rham cohomology groups* of the smooth manifold M.

Example 8.13

$$H^p(\text{point}) = \begin{cases} \mathbb{R} & p = 0 \\ 0 & p \neq 0 \end{cases}$$

Example 8.14 Let M be any connected manifold. Suppose we have a smooth function $f \in C^\infty(M)$ such that $df = 0$. Then, in terms of local coordinates $(x^1, \ldots, x^n)$, we know that $\partial f/\partial x^i = 0$. Hence the function f is constant.

By definition of de Rham cohomology, it follows that $H^0(M) \cong \mathbb{R}$.

Example 8.15 Let M and N be smooth manifolds. Then $H^p(M \coprod N) = H^p(M) \oplus H^p(N)$ for all p.

In particular, using the above example, for any smooth manifold M, we have an isomorphism $H^0(M) \cong \mathbb{R}^{\pi_0(M)}$.

To work out the de Rham cohomology groups of more complicated examples, and to see why they are interesting, we need to develop the general theory.

Proposition 8.16 *Let $f\colon M \to N$ be a smooth map of manifolds. Then there is a contravariantly functorial induced chain map of cochain complexes $f^\star\colon \Omega^\star(N) \to \Omega^\star(M)$.*

Proof: Let $\omega \in \Omega^p(N)$. Choose local coordinates $(x^1, \ldots, x^m)$ on the manifold M and write $f(x^1, \ldots, x^n) = (y^1, \ldots, y^n)$. Write:

$$\omega = \sum_{i_1 < \cdots < i_p} g_{i_1, \ldots, i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

Then we can define a differential form $f^\star(\omega) \in \Omega^p(M)$ by the formula:

$$f^\star \omega = \sum_{\substack{i_1 < \cdots < i_p \\ j_1 < \cdots < j_p}} g_{i_1, \ldots, i_p} \circ f \frac{\partial}{\partial x^{i_1}} dy^{j^1} \cdots \frac{partial x^{i_k}}{\partial y^{j_k}} dy^{j_1} \wedge \cdots \wedge dy^{j_k}$$

It is easy to check that $f^\star(d\omega) = d(f^\star \omega)$. Hence the map $f^\star$ is a chain map. Functoriality is also easy to check. □

We thus obtain contravariantly functorial induced maps of de Rham cohomology groups:

$$f^\star\colon H^p(N) \to H^p(M)$$

Lemma 8.17 *Let M be a smooth manifold. Define maps $s\colon M \to M \times \mathbb{R}$ and $\pi\colon M \times \mathbb{R} \to M$ by the formulae:*

$$s(x) = (x, 0) \qquad \pi(x, t) = x$$

respectively. Then the induced maps $s^\star\colon H^p(M \times \mathbb{R}) \to H^p(M)$ and $\pi^\star\colon H^p(M) \to H^p(M \times \mathbb{R})$ are isomorphisms.

Proof: Observe that $\pi \circ s = 1_M$ so the composition $s^\star \circ \pi^\star$ is the identity map. We aim to construct a chain homotopy between the map $\pi^\star \circ s^\star\colon \Omega^p(M \times \mathbb{R}) \to \Omega^p(M \times \mathbb{R})$ and the identiy.

Let $\omega \in \Omega^p(M \times \mathbb{R})$. Then we can write:

$$\omega = f(x,t)\pi^\star(\eta) + g(x,t)\pi^\star(\theta) \wedge dt$$

where $\eta \in \Omega^p(M)$ and $\theta \in \Omega^{p-1}(M)$. Define a form $K\omega \in \Omega^{p-1}(M \times \mathbb{R})$ by writing:

$$K\omega(x,t) = \pi^\star(\theta)(x,t) \int_0^t g(x,s)\ ds$$

A fairly long but perfectly straightforward calculation tells us that:

$$\omega - \pi^\star \circ s^\star \omega = (-1)^{p-1}(dK\omega - Kd\omega)$$

so we have the desired chain homotopy. □

Theorem 8.18 *Let $f, g: M \to N$ be smoothly homotopic maps. Then the induced maps $f^\star, g^\star: H^p(N) \to H^p(M)$ are equal.*

Proof: We have a map $F: M \times \mathbb{R} \to N$ such that $F(-,t) = f$ whenever $t \leq 0$ and $F(-,t) = g$ whenever $t \geq 1$. Define maps $s_0, s_1: M \to M \times \mathbb{R}$ by the formulaee:

$$s_0(x) = (x,0) \qquad s_1(x) = (x,1)$$

By the above lemma the projection $\pi: M \times \mathbb{R} \to M$ induces an isomorphism $\pi^\star: H^p(M) \to H^p(M \times \mathbb{R})$. The maps $s_0^\star$ and $s_1^\star$ are both inverses of the isomorphism $\pi^\star$ and therefore equal. Hence the maps $f^\star = s_0^\star \circ F^\star$ and $g^\star = s_1^\star \circ F^\star$ are equal. □

Corollary 8.19 *The assignment of homotopy groups, $M \mapsto H^p(M)$, is a contravariant functor from the category of smooth manifolds and* continuous *maps to the category of abelian groups. The functorially induced maps agree with those defined above in the smooth case.*

Further, if $f, g: M \to N$ are homotopic maps, then the induced maps $f^\star, g^\star: H^p(N) \to H^p(M)$ are equal.

Proof: Let $f: M \to N$ be a smooth map. By theorem 7.7 there is a smooth map, $\tilde{f}: M \to N$ that is homotopic to f. We try to define our induced map $f^\star: H^p(N) \to H^p(M)$ by the formula $f^\star = \tilde{f}^\star$.

Suppose that $\tilde{f}'$ is another smooth map that is homotopic to f. Then the maps $\tilde{f}'$ and $\tilde{f}$ are homotopic. It follows by corollary 7.8 that the maps $\tilde{f}'$ and $\tilde{f}$ are smoothly homotopic, and so $\tilde{f}'^\star = \tilde{f}^\star$, and the map $f^\star$ is well-defined.

The proof that homotopic maps $f, g: M \to N$ induce the same homomorphisms $f^\star, g^\star: H^p(N) \to H^p(M)$ is proved similarly. □

The following consequence of homotopy-invariance is sometimes called the *Poincarè lemma*.

Corollary 8.20

$$H^p(\mathbb{R}^n) = \begin{cases} \mathbb{R} & p = 0 \\ 0 & p \neq 0 \end{cases}$$

□

Lemma 8.21 *Let M be a smooth manifold, and let U and V be open submanifolds such that $M = U \cup V$. Let $i\colon U \cap V \hookrightarrow U$, $j\colon U \cap V \hookrightarrow V$, $k\colon U \hookrightarrow M$, and $l\colon V \hookrightarrow M$ be the various inclusion maps. Then we have a short exact sequence of cochain complexes:*

$$0 \longrightarrow \Omega^\star(M) \stackrel{(k^\star, l^\star)}{\longrightarrow} \Omega^\star(U) \oplus \Omega^\star(V) \stackrel{i^\star - j^\star}{\longrightarrow} \Omega^\star(U \cap V) \longrightarrow 0$$

Proof: The only remotely awkward part in proving exactness of the given sequence is proving that the map $i^\star - j^\star$ is surjective.

Let $\omega \in \Omega^p(U \cap V)$. Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to the open cover $\{U, V\}$. Then we have forms $\rho_V \omega \in \Omega^p(U)$ and $\rho_U \omega \in \Omega^p(V)$. We can write:

so the map $i^\star - j^\star$ is surjective as required. □

By the snake lemma, the above result implies the following theorem.

Theorem 8.22 *There is a natural map $\partial\colon H^p(U \cap V) \to H^{p+1}(M)$ such that we have a long exact sequence:*

$$\longrightarrow H^p(M) \stackrel{(k^\star, l^\star)}{\longrightarrow} H^p(U) \oplus H^p(V) \stackrel{i^\star - j^\star}{\longrightarrow} H^p(U \cap V) \stackrel{\partial}{\longrightarrow} H^{p+1}(M) \longrightarrow$$

□

The above long exact sequence is called the *Mayer-Vietoris sequence* in de Rham cohomology. It is an extremely useful tool for calculations.

Example 8.23 The sphere, S^{n+1}, can be expressed as a union $S^{n+1} = U \cup V$ where the sets U and V are contractible, and the intersection, $U \cap V$, is homotopy-equivalent to the sphere S^n.

The Mayer-Vietoris sequence thus takes the following form

$$\longrightarrow H^p(S^{n+1}) \stackrel{(k^\star, l^\star)}{\longrightarrow} H^p(+) \oplus H^p(+) \stackrel{i^\star - j^\star}{\longrightarrow} H^p(S^n) \stackrel{\partial}{\longrightarrow} H^{p+1}(S^{n+1}) \longrightarrow$$

where $+$ is the one point space.

Let $p > 0$. Then $H^p(+) = 0$ and $H^{p+1}(+) = 0$ so the map $\partial\colon H^p(S^n) \to H^{p+1}(S^{n+1})$ is an isomorphism.

If $n > 0$ and $p = 0$, the above exact sequence takes the form

$$0 \longrightarrow \mathbb{R} \stackrel{\alpha}{\longrightarrow} \mathbb{R} \oplus \mathbb{R} \stackrel{\beta}{\longrightarrow} \mathbb{R} \stackrel{\partial}{\longrightarrow} H^1(S^{n+1}) \longrightarrow 0$$

where the maps α and β are defined by the formulae

$$\alpha(x) = (x, x) \qquad \beta(x, y) = x - y$$

respectively. The image of the map β is the entire set $\mathbb{R}$, so by exactness, the map ∂ is zero, and the group $H^1(S^{n+1})$ is zero. Since we have an isomorphism $H^p(S^n) \cong H^{p+1}(S^{n+1})$, we know that $H^p(S^n) = 0$ if $n > p$, and $p > 0$.

The 'sphere' S^0 is just a set containing two points. Thus, if $n = 0$ and $p > 0$ the above exact sequence becomes

$$0 \longrightarrow H^{p+1}(S^1) \longrightarrow 0 \longrightarrow 0$$

and, using the isomorphism $H^p(S^n) \cong H^{p+1}(S^{n+1})$, see that $H^p(S^n) = 0$ if $p > n$.

Finally, if $n = 0$ and $p = 0$ we have the sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{\alpha} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\beta} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\partial} H^1(S^{n+1}) \longrightarrow 0$$

where the maps α and β are defined by the formulae

$$\alpha(x) = (x, x) \qquad \beta(x, y) = (x - y, x - y)$$

respectively. The map β has image

$$\{(x, x) \in \mathbb{R} \oplus \mathbb{R}\}$$

So we have a short exact sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{\gamma} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\partial} H^1(S^{n+1}) \longrightarrow 0$$

where $\gamma(x) = (x, x)$.

We can define a homomorphism $\gamma' \colon \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}$ by the formula $\gamma'(x, y) = x$. Then $\gamma'\gamma = 1_{\mathbb{R}}$ so by proposition **??** the groups $\mathbb{R} \oplus \mathbb{R}$ and $\mathbb{R} \oplus H^1(S^1)$ are isomorphic. It follows that the group $H^1(S^1)$ is isomorphic to $\mathbb{R}$. Using the isomorphism $H^p(S^n) \cong H^{p+1}(S^{n+1})$, we see that $H^n(S^n) = \mathbb{R}$ if $n > 0$.

To summarise:

$$H^p(S^n) = \begin{cases} \mathbb{R} & p = n,\ p = 0 \\ 0 & \text{otherwise} \end{cases}$$

The above example tells us that the spheres S^m and S^n are not homotopy-equivalent, and in particular not homeoemorphic if $m \neq n$. Therefore the spaces $S^m \backslash \{+\}$ and $S^n \backslash \{+\}$ obtained by deleting a single point from these spheres are not homeomorphic. But the spaces $S^m \backslash \{+\}$ and $S^n \backslash \{+\}$ are homeomorphic to the spaces $\mathbb{R}^m$ and $\mathbb{R}^m$ respectively, and so the spaces $\mathbb{R}^m$ and $\mathbb{R}^m$ are not homeomorphic.

We will see some other fundamental geometric results proved through similar computations in the exercises and following chapters.

We now turn our attention to compactly supported differential forms. We begin by noting that the differential of a compactly supported form is compactly supported so we have a cochain complex:

$$0 \to \Omega_c^0(M) \xrightarrow{d} \Omega_c^1(M) \xrightarrow{d} \Omega_c^2(M) \xrightarrow{d} \cdots$$

The cohomology groups of this complex are the *compactly supported de Rham cohomology groups*, $H_c^p(M)$. Similar calculations to those performed above yield the following results:

Proposition 8.24

$$H_c^p(\mathbb{R}^n) = \begin{cases} \mathbb{R} & p = n \\ 0 & p \neq n \end{cases}$$

□

Compactly supported cohomology also posesses a 'wrong way functoriality' property. To be specific, if M is a manifold and $U \subseteq M$ is an open submanifold, then any compactly supported form $\omega \in \Omega^p_c(U)$ can be extended (by zero) to a compactly supported form $i(\omega) \in \Omega^p_c(M)$. The inclusion map $i: U \hookrightarrow M$ thus induces a map of homology groups $i_\star: H^p_c(U) \to H^p_c(M)$.

Mayer-Vietoris sequences also function in the compactly supported setting.

Theorem 8.25 *Let M be a smooth manifold, and let U and V be open submanifolds such that $M = U \cup V$. Let $i: U \cap V \hookrightarrow U$, $j: U \cap V \hookrightarrow V$, $k: U \hookrightarrow M$, and $l: V \hookrightarrow M$ be the various inclusion maps. Then there is a natural map $\partial: H^p_c(U \cap V) \to H^{p+1}_c(M)$ such that we have a long exact sequence:*

$$\longrightarrow H^p_c(U \cap V) \stackrel{(i_\star, j_\star)}{\longrightarrow} H^p_c(U) \oplus H^p_c(V) \stackrel{k_\star - l_\star}{\longrightarrow} H^p_c(M) \stackrel{\partial}{\longrightarrow} H^{p-1}_c(U \cap V) \longrightarrow$$

□

9 Orientation and Duality

9.1 Orientations on Vector Bundles

Let $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$ be bases for a real vector space V. We can form an invertible linear transformation $T: V \to V$ by writing $Te_i = e'_i$. There is a matrix, A, associated to T; the entries a^{ij} are defined by the formula

$$e'_i = \sum_j a^{ij} e_j$$

Since the linear transformation T is invertible, the matrix A has non-zero determinant.

Definition 9.1 The bases $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$ are said to have the *same orientation* if the determinant $\det(A)$ is positive. Otherwise, they are said to have the *opposite orientation*.

Thus an orientation of a vector space is a certain equivalence class of bases. A vector space has precisely two possible orientations. An *oriented vector space* is simply a vector space equipped with a choice of orientation; we say a basis in the relevant equivalence class is one that defines the orientation.

Definition 9.2 Let V and V' be oriented vector spaces of the same dimension. Let $\{e_1, \ldots, e_n\}$ be a basis for the vector space V defining the given orientation. Then an invertible linear transformation $T: V \to V'$ is said to be *orientation-preserving* if the basis $\{Te_1, \ldots, Te_n\}$ defines the orientation given for the vector space V'.

It is easy to check that the above definition does not depend on the exact choice of basis $\{e_1, \ldots, e_n\}$ with the given orientation. An invertible linear transformation that is not orientation-preserving is termed *orientation-reversing*.

The *standard orientation* for the vector space $\mathbb{R}^n$ is that defined by the standard basis

$$\{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\}$$

Definition 9.3 Let E be a real vector bundle over a space X. An *orientation* for E is a choice of orientation for each vector space E_x such that if $U \subseteq X$ is a connected open set, and $\psi\colon E_U \to U \times \mathbb{R}^n$ is a bundle isomorphism, the linear transformations of fibres

$$\psi\colon E_x \to \{x\} \times \mathbb{R}^n$$

are either all orientation-preserving or orientation-reversing.

A real vector bundle equipped with an orientation is called *oriented*. A real vector bundle that *can* be equipped with an orientation is called *orientable*.

Example 9.4 Any trivial real vector bundle is orientable.

The following result is an exercise in understanding the definition.

Proposition 9.5 *Let E be a real orientable vector bundle. Then E can be equipped with precisely two possible orientations.* □

Definition 9.3 is perhaps the best definition to work with when proving facts about oriented vector bundles or trying to prove that a given vector bundle is *not oriented*. However, when trying to define orientations on vector bundles, a reformulation is useful.

Proposition 9.6 *Let E be a real vector bundle over a space X equipped with an orientation on each fibre E_x. Suppose we have an atlas of local trivialisations*

$$\{(U_\lambda, \psi_\lambda) \mid \lambda \in \Lambda\}$$

such that for each $\lambda \in \Lambda$ the linear transformations

$$\psi_\lambda\colon E_x \to \{x\} \times \mathbb{R}^n$$

are either all orientation-preserving or orientation-reversing.

Then the bundle E is oriented. □

Example 9.7 Let E be a *complex* vector bundle over a space X, that is to say a vector bundle where the fibres are isomorphic to $\mathbb{C}^n$ for some n. There is an underlying real vector bundle $E_\mathbb{R}$; the fibre $(E_\mathbb{R})_x$ is defined by considering the fibre E_x as a real rather than a complex vector space. We claim that the bundle $E_\mathbb{R}$ is orientable.

To see this, let $\{(U_\lambda, \psi_\lambda) \mid \lambda \in \Lambda\}$ be an atlas of local trivialisations for the complex vector bundle E. Let $\{e_1, \ldots, e_n\}$ be a basis for the complex vector space E_x. Then the real vector space $(E_\mathbb{R})_x$ has a basis

$$\{e_1, ie_1, e_2, ie_2, \ldots, e_n, ie_n\}$$

Let us equip the space $(E_\mathbb{R})_x$ with the orientation defined by the above basis. Let $\{v_1, \ldots, v_{2n}\}$ be the standard basis for the real vector space $\mathbb{R}^{2n}$, and let $\{w_1, \ldots, w_n\}$ be the standard basis for the complex vector space $\mathbb{C}^n$. We have an isomorphism $T\colon \mathbb{C}^n \to \mathbb{R}^{2n}$ of *real* vector spaces defined by writing

$$Tw_j = v_{2j-1} \qquad T(iw_j) = v_{2j}$$

The real vector bundle $\mathbb{E}_{\mathbb{R}}$ has an atlas of local trivialisations $\{(U_\lambda, T \circ \psi_\lambda) \mid \lambda \in \Lambda\}$. It is easy to check that the maps

$$T \circ \psi_\lambda \colon (\mathbb{E}_{\mathbb{R}})_x \to \{x\} \times \mathbb{R}^{2n}$$

are all orientation-preserving.

The argument used in the above example tells us not only that the real vector bundle $E_{\mathbb{R}}$ is orientable, but gives us a canonical orientation. This observation will be important in later chapters when we explore the theory of characteristic classes.

Example 9.8 Consider the Möbius band, E, as a one-dimensional real vector bundle over the circle S^1. Consider the space

$$A = \frac{[0,1] \times \{-1, 1\}}{\sim}$$

where $[0,t] \sim [1,-t]$.

Suppose that the Möbius band can be equipped with an orientation. Then for each point x there is a vector $s(x) \in A \cap E_x$ of norm one such that the basis $\{s(x)\}$ has the given orientation.

According to the definition of an orientation, we have constructed a nowhere-zero section $s \colon S^1 \to E$. But according to proposition 5.2 this would prove that the Möbius band is trivial, which is certainly not the case.

Thus the Möbius band is not orientable.

9.2 Orientations of Manifolds

Definition 9.9 Let M be a smooth manifold. Then M is termed *orientable* if the tangent bundle TM is orientable.

An *orientation* of an orientable manifold is a choice of orientation for the tangent bundle TM.

Example 9.10 Any manifold with trivial tangent bundle is orientable.

Manifolds with trivial tangent bundles are called *parallelisable*.

Example 9.11 The Möbius band is not an orientable manifold. The argument to prove this is similar to that of example 9.8.

Let $f \colon M \to N$ be a diffeomorphism. Then the differential $(Df)_x \colon T_xM \to T_{f(x)}N$ is an isomorphism. If M and N are oriented manifolds, the diffeomorphism f is termed *orientation-preserving* if the isomorphisms $(Df)_x \colon T_xM \to T_{f(x)}N$ are all orientation-preserving, and *orientation-reversing* if the isomorphisms $Df \colon T_xM \to T_{f(x)}N$ are all orientation-reversing.

Definition 9.12 Let M be a manifold. A smooth atlas $\{(U_\lambda, V_\lambda, \phi_\lambda)\}$ for M is said to be *oriented* if the diffeomorphisms $(\phi_\mu^{-1} \circ \phi_\lambda)|_{\phi_\lambda^{-1}[V_\lambda \cap V_\mu]}$ are all orientation-preserving.

A smooth manifold is said to be *oriented* if it is equipped with an oriented atlas. Disentangling the relevant definitions gives us the following result.

Proposition 9.13 *An orientation on a smooth manifold determines, and is determined by, a choice of oriented atlas.* □

The above lets us easily see that, for example, the sphere S^n is orientable.

Note that our work on orientations all also makes sense for manifolds with boundary. Further, we have the following.

Proposition 9.14 *Let M be an oriented manifold, with boundary ∂M. Then the boundary has an orientation induced from that of M.* □

Orientation can be reformulated in terms of differential forms. To prove the reformulation is valid, we need the following lemma. The proof is a straightforward calculation based on the definition of a determinant.

Lemma 9.15 *Let (U, V, ϕ) and $(\tilde{U}, \tilde{V}, \tilde{\phi})$ be charts, and let $\phi(x^1, \ldots, x^n) = \tilde{\phi}(\tilde{x}^1, \ldots, \tilde{x}^n)$*

Let J be the Jacobian determinant

$$J = \det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right)$$

Then

$$dx^1 \wedge \cdots \wedge dx^n = J d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n$$

□

Proposition 9.16 *Let M be a smooth n-dimensional manifold. Then the manifold M is orientable if and only if there is a differential form $\omega \in \Omega^n(M)$ such that $\omega(x) \neq 0$ for all $x \in M$.*

Proof: Suppose the manifold M is orientable. Then we have an oriented locally finite atlas

$$\{(U_\lambda . V_\lambda, \phi_\lambda) \mid \lambda \in \Lambda\}$$

Let $\phi_\lambda(x_\lambda^1, \ldots, x_\lambda^n) = \phi_\mu(x_\mu^1, \ldots, x_\mu^n)$. Then the determinants

$$\det\left(\frac{\partial x_\lambda^i}{\partial x_\mu^j}\right)$$

are all positive.

Choose a smooth partition of unity, $\{\rho_\lambda \mid \lambda \in \Lambda\}$, subordinate to the locally finite cover $\{U_\lambda \mid \lambda \in \Lambda\}$. Define

$$\omega = \sum_{\lambda \in \Lambda} \rho_\lambda dx_\lambda^1 \wedge \cdots \wedge dx_\lambda^n$$

By lemma 9.15 and positivity of the determinants arising form coordinate changes, it follows that $\omega(x) \neq 0$ for every point $x \in M$.

Conversely, suppose we have a nowhere-zero differential form $\omega \in \Omega^n(M)$. Then for each point $x \in M$ we have local coordinates $(x^1, \dots, x^n)$ such that

$$\omega = f dx^1 \wedge \cdots \wedge dx^n$$

for some non-vanishing smooth function f. By permuting two of the coordinates (x^i) if necessary, we can assume that the function f is strictly positive.

If $(\tilde{x}^1, \dots, \tilde{x}^n)$ is another set of such local coordinates, we can write

$$\omega = \tilde{f} d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n$$

where the function $\tilde{f}$ is also strictly positive. Let J be the Jacobian determinant $\det(\partial x^i / \partial \tilde{x}^j)$. Then by lemma 9.15, $J = \tilde{f}/f$. Hence the determinant J is positive.

Thus the collection of local coordinate systems we have defined gives us an oriented atlas for M. □

Definition 9.17 Let M be a Riemannian manifold equipped with some oriented atlas. In local coordinates $(x^1, \dots, x^n)$ let us write

$$g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$$

and let $|g|$ denote the determinant of the matrix (g_{ij}). Then we define the *volume form* on M by the formula

$$vol = \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n$$

The volume form does not depend on a choice of local coordinates.

9.3 Integration

Suppose that M is a smooth n-dimensional manifold, equipped with an oriented atlas. Let $[M]$ denote the orientation defined by the atlas. Consider a compactly supported differential form $\omega \in \Omega^n_c(M)$. Let (U, V, ϕ) be a chart, and $(x^1, \dots x^n) \in U$. Then we can write

$$\omega(x) = f(x) dx^1 \wedge \cdots \wedge dx^n$$

when $x \in V$.

Definition 9.18 If $\omega(x) = 0$ for $x \notin V$, we define the *integral*:

$$\int_{[M]} \omega = \int f(\phi(x^1, \dots, x^n)) \; dx^1 \dots dx^n$$

Suppose we have another chart with the same orientation, $(\tilde{U}, \tilde{V}, \tilde{\phi})$, and have $\phi(x^1, \dots, x^n) = \tilde{\phi}(\tilde{x}^1, \dots, \tilde{x}^n)$. Then the determinant $J(\tilde{x}^1, \dots, \tilde{x}^n) = \det(\partial x^i / \partial \tilde{x}^j)$ is positive, and we have the formula

$$f dx^1 \wedge \cdots \wedge dx^n = J f d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n$$

by lemma 9.15. But by the change of variables formula for integration:

$$\int f\phi(x^1,\ldots,x^n)\ dx^1\cdots dx^n = \int J(\tilde{x}^1,\ldots\tilde{x}^n)f\tilde{\phi}(\tilde{x}^1,\ldots,\tilde{x}^n)\ d\tilde{x}^1\cdots d\tilde{x}^n$$

so the integral of the differential form ω does not depend on a choice of local coordinates. More generally, the following definition makes sense.

Definition 9.19 Let M be an n-dimensional manifold, with oriented locally finite atlas $\{(U_\lambda, V_\lambda, \phi_\lambda) \mid \lambda \in \Lambda\}$. Let $\{\rho_\lambda \mid \lambda \in \Lambda\}$ be a partition of unity subordinate to the open cover $\{U_\lambda \mid \lambda \in \Lambda\}$, and let $[M]$ be the orientation defined by the atlas. Let $\omega \in \Omega^n_c(M)$ be a compactly supported n-form. Then we define the *integral*:

$$\int_{[M]} \omega = \sum_{\lambda\in\Lambda} \int \rho_\lambda \omega$$

It is easy to check that the above definition does not depend on choice of appropriate oriented atlas or on the partition of unity chosen.

Lemma 9.20 *Let $\omega \in \Omega^{n-1}_c(\mathbb{R}^n)$ be a compactly supported $(n-1)$-form. Then:*

$$\int_{\mathbb{R}^n} d\omega = 0$$

Proof: Without loss of generality, assume that $\omega = f dx^2 \wedge \cdots \wedge dx^n$. Then:

$$d\omega = \frac{\partial f}{\partial x^1} dx^1 \wedge \cdots \wedge dx^n$$

By Fubini's theorem:

$$\int_{\mathbb{R}^n} d\omega = \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f}{\partial x^1}\ dx^1 \right)\ dx^2 \ldots dx^n$$

But

$$\int_{-\infty}^{\infty} \frac{\partial f}{\partial x^1}\ dx^1 = 0$$

by the fundamental theorem of calculus and the result follows. □

A similar argument gives us the following result for integrals over half-space.

Lemma 9.21 *Let $\omega \in \Omega^{n-1}_c(\mathbb{R}^n_+)$ be a compactly supported $(n-1)$-form. Then:*

$$\int_{\mathbb{R}^n} d\omega = \int_{\mathbb{R}^{n-1}} \omega$$

□

Theorem 9.22 *Stokes' theorem*
Let M be a smooth oriented bounded manifold of dimension n. Let $\omega \in \Omega^{n-1}_c(M)$ be a differential $(n-1)$-form. Then:

$$\int_{[M]} d\omega = \int_{[\partial M]} \omega$$

Proof: Let $(U_\lambda, V_\lambda, \phi_\lambda)$ be an oriented atlas. Let $\{\rho_\lambda \mid \lambda \in \Lambda\}$ be a partition of unity subordinate to the open cover $\{U_\lambda \mid \lambda \in \Lambda\}$. Then:

$$\int_{[M]} d\omega = \sum_{\lambda\in\Lambda} \int_{[M]} \rho_\lambda d\omega$$

Without loss of generality, assume that the open set V_λ is diffeomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^n_+$. If the set V_λ is diffeomorphic to the space $\mathbb{R}^n$ then:

$$\int_{[M]} \rho_\lambda d\omega = 0$$

by lemma 9.20. If the set V_λ is diffeomorphic to the space $\mathbb{R}^n_+$ then:

$$\int_{[M]} \rho_\lambda d\omega = \int_{[\partial M]} \rho_\lambda \omega$$

by lemma 9.21. Hence:

$$\sum_{\lambda\in\Lambda} \int_{[M]} \rho_\lambda d\omega = \sum_{\lambda\in\Lambda} \int_{[\partial M]} \rho_\lambda \omega$$

and we are done. □

Corollary 9.23 *Let M be a smooth oriented manifold without boundary. Then we have a well-defined linear map*

$$\int_{[M]} : H^n_c(M) \to \mathbb{R}$$

defined by the formula

$$\int_{[M]} [\omega] = \int_{[M]} \omega.$$

Proof: Let $\eta \in \Omega^{n-1}_c(M)$. Then by Stokes' theorem

$$\int_{[M]} d\eta = \int_{\emptyset} \eta = 0.$$

Hence, if $[\omega] = [\omega']$, then $\omega - \omega' = d\eta$, so

$$\int_{[M]} \omega = \int_{[M]} \omega'.$$

The result now follows. □

9.4 Poincaré Duality

Let M be a smooth manifold. Let $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(N)$. Then we have the exterior product $\omega \wedge \eta \in \Omega^{p+q}(M)$. By proposition 8.10 we know that

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta.$$

This gives us the following result.

Proposition 9.24 *We have well-defined linear maps*

$$\cup \colon H^p(M) \times H^q(M) \to H^{p+q}(M)$$

and

$$\cup \colon H^p(M) \times H^q_c(M) \to H^{p+q}_c(M)$$

both defined by the formula

$$[\omega] \cup [\eta] = [\omega \wedge \eta].$$

□

We call the element $[\omega] \cup [\eta]$ the *cup product* of $[\omega]$ and $[\eta]$. The cup product is easily seen to be associative.

Suppose M is n-dimensional. Then by the above we have a bilinear map

$$\langle -, - \rangle \colon H^p(M) \times H^{n-p}_c(M) \to \mathbb{R}$$

defined by the formula

$$\langle [\omega], [\eta] \rangle = \int_{[M]} \omega \wedge \eta.$$

Definition 9.25 We say a bilinear map $B \colon V \times W \to \mathbb{R}$ is *non-degenerate* if $B(v, w) = 0$ for all $v \in V$ implies $w = 0$.

It is equivalent to the above to say that B is non-degenerate if $B(v, w) = 0$ for all $w \in W$ implies $v = 0$. Given a bilinear map B, we have a linear map $A \colon V \to W^*$ defined by the formula

$$A(v)(w) = B(v, w).$$

The bilinear map B is non-degenerate if and only if the linear map A is an isomorphism.

Poincaré duality is the assertion that the bilinear map $\langle -, - \rangle \colon H^p(M) \times H^{n-p}_c(M) \to \mathbb{R}$ is non-degenerate.

The key part of the proof is the following lemma.

Lemma 9.26 *Let $M = U \cup V$, where U and V are open. Then the following diagram is commutative up to a + or − sign:*

$$\begin{array}{ccccccccc}
\longrightarrow & H^p(M) & \xrightarrow{(k^\star, l^\star)} & H^p(U) \oplus H^p(V) & \xrightarrow{i^\star - j^\star} & H^p(U \cap V) & \xrightarrow{\partial} & H^{p+1}(M) & \longrightarrow \\
 & \otimes & & \otimes & & \otimes & & \otimes & \\
\longleftarrow & H^p_c(M) & \xleftarrow{k_\star - l_\star} & H^p_c(U) \oplus H^p_c(V) & \xleftarrow{(i_\star, j_\star)} & H^p_c(U \cap V) & \xleftarrow{\partial} & H^{p-1}_c(M) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & \mathbb{R} & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} &
\end{array}$$

Here, the top rows are Mayer-Vietoris sequences in de Rham and compactly supported de Rham cohomology respectively, and the vertical arrows are defined by the map $\langle -, - \rangle$.

Proof: Let $\eta \in \Omega^p(M)$, $\xi_1 \in \Omega_c^{n-p}(U)$, $\xi_2 \in \Omega_c^{n-p}(V)$. Then

$$\langle \eta, (k_* - l_*)(\xi_1, \xi_2) \rangle = \langle \eta, \xi_1 \rangle + \langle \eta, \xi_2 \rangle$$

and

$$\langle (k^*, l^*(\eta), (\xi_1, \xi_2) \rangle = \langle (\eta, \eta), (\xi_1, \xi_2) \rangle = \langle \eta, \xi_1 \rangle + \langle \eta, \xi_2 \rangle$$

so the first square commutes. The second square commutes similarly.

Let $\eta \in \Omega^p(U \cap V)$ and $\xi \in \Omega_x^{n-p-1}(M)$. Then

$$\langle \eta, d\xi \rangle = \int_{[M]} \eta \wedge d\xi.$$

Recall from corollary 8.10 that

$$d(\eta \wedge \xi) = d\eta \wedge \xi + (-1)^p \eta \wedge d\xi$$

and by Stokes' theorem

$$\int_{[M]} d(\eta \wedge \xi) = 0.$$

Hence

$$\int_{[M]} \eta \wedge d\xi = \pm \int_{[M]} d\eta \wedge \xi = \langle d\eta, \xi \rangle$$

and we are done. □

Definition 9.27 Let M be a smooth manifold. An open cover $\{U_\lambda \mid \lambda \in \Lambda\}$ is called a *good cover* of all non-empty finite intersections $U_{\lambda_1} \cap \cdots \cap U_{\lambda_n}$ are diffeomorphic to $\mathbb{R}^n$.

The proof of the following is straightforward.

Proposition 9.28 *Every manifold (without boundary) has a good cover, and every compact manifold has a finite good cover.*

We say a manifold is of *finite type* if it has a finite good cover.

Theorem 9.29 (Poincaré Duality) *Let M be an oriented manifold of finite type. Then the bilinear map*

$$\langle -, - \rangle \colon H^p(M) \times H_c^{n-p}(M) \to \mathbb{R}$$

defined by the formula

$$\langle [\omega], [\eta] \rangle = \int_{[M]} \omega \wedge \eta$$

is non-degenerate.

Proof: Recall, from the Poincaré lemmas

$$H^p(\mathbb{R}^n) \equiv \begin{cases} \mathbb{R} & p = 0 \\ 0 & \text{otherwise} \end{cases} \qquad H^p_c(\mathbb{R}^n) \equiv \begin{cases} \mathbb{R} & p = n \\ 0 & \text{otherwise} \end{cases}$$

Suppose $p = 0$. Then $\eta \in \Omega^0(\mathbb{R}^n)$ means $\eta \in C(\mathbb{R}^n)$, and $\xi \in \Omega^n_c(\mathbb{R}^n)$ means $\xi = g dx^1 \cdots dx^n$ where $g \in C_c(\mathbb{R}^n)$. So

$$\langle [\eta], [\xi] \rangle = \int_{\mathbb{R}^n} \eta(x^1, \ldots, x^n) g(x^1, \ldots, x^n)\ dx^1 \cdots dx^n$$

Suppose $\langle [\eta], [\xi] \rangle = 0$ for all $\xi \in \Omega^n_c(\mathbb{R}^n)$. Then by the fact that the above integral defines an inner product on $\mathbb{R}^n$, it follows that $g = 0$, and so $\eta = 0$. Therefore the bilinear map $\langle -, - \rangle \colon H^0(M) \times H^n_c(M) \to \mathbb{R}$ is non-degenerate.

Similarly, the product $\langle -, - \rangle \colon H^n(\mathbb{R}^n) \times H^0_c(\mathbb{R}^n) \to \mathbb{R}$ is non-degenerate. If $p \neq 0, n$, then $H^p(\mathbb{R}^n) = 0$ and $H^{n-p}_c(\mathbb{R}^n) = 0$, so the bilinear map $\langle -, - \rangle \colon H^p(\mathbb{R}^n) \times H^{n-p}_c(\mathbb{R}^n) \to \mathbb{R}$ has to be non-degenerate.

Now, pick a finite good cover $\{U_1, \ldots, U_m\}$ of M. Then by the above, the pairings

$$\langle -, - \rangle \colon H^p(U_i) \times H^{n-p}_c(U_i) \to \mathbb{R}$$

and

$$\langle -, - \rangle \colon H^p(U_i \cap U_j) \times H^{n-p}_c(U_i \cap U_j) \to \mathbb{R}$$

are non-degenerate, meaning $H^p(U_i) \equiv H^{n-p}_c(U_i)^*$ and $H^p(U_i \cap U_j) \equiv H^{n-p}_c(U_i \cap U_j)^*$.

By lemma 9.26, we have a commutative diagram

$$\begin{array}{ccccccccc} H^{p-1}(U_i) \oplus H^{p-1}(U_j) & \to & H^{p-1}(U_i \cap U_j) & \to & H^p(U_i cup U_j) & \to & H^p(U_i) \oplus H^p(U_j) & \to & H^p(U \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ H^{p-1}_c(U_i)^* \oplus H^{p-1}_c(U_j)^* & \to & H^{p-1}_c(U_i \cap U_j)^* & \to & H^p_c(U_i cup U_j)^* & \to & H^p_c(U_i)^* \oplus H^p_c(U_j)^* & \to & H^p_c(U \end{array}$$

where the left two and right two vertical maps are isomorphisms. By the five lemma, the maps $H^p(U_i \cup U_j) \to H^{n-p}_c(U_i \cup U_j)^*$ are all isomorphisms.

Repeating this process, we see that for every finite union $X = U_{r_1} \cup \cdots \cup U_{r_s}$, the map $H^p(X) \to H^p_c(X)^*$ is an isomorphism. In particular, as $M = U_1 \cup \cdots \cup U_m$, we have that the map $H^p(M) \to H^{n-p}_c(M)^*$ is an isomorphism, ie: that the bilinear map

$$\langle -, - \rangle \colon H^p(M) \times H^{n-p}_c(M) \to \mathbb{R}$$

is non-degenerate. □

Corollary 9.30 *Let M be a compact orientable manifold. Then*

$$H^p(M) \cong M^{n-p}(M)^*.$$

□

In particular, it follows that $H^n(M) = \mathbb{R}$ when M is an n-dimensional compact orientable manifold. This, together with the fact that $H^p(M) = 0$ if $p > n$ tells us that the dimension of a compact orientable manifold is a homotopy-invariant.