Mathematics for Physicists

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1 Vectors and Geometry

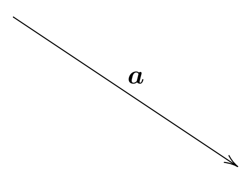
1.1 The Basics

- A *scalar* is a quantity which only has magnitude, for example mass, time, temperature.
- A *vector* has both a direction and a magnitude, for example, velocity, force, electric field.

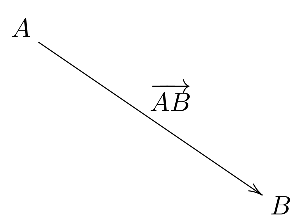
Two vectors are equal if both their direction and their magnitude is equal.

We denote scalars with ordinary letters, for example m, t, T. We denote vectors with bold letters, for example $\boldsymbol{v}, \boldsymbol{F}, \boldsymbol{E}$. In handwritten work, bold letters are hard to reproduce, so we underline instead: $\underline{v}, \underline{F}, \underline{E}$. Some people write vectors with arrows above them: $\overrightarrow{v}, \overrightarrow{F}, \overrightarrow{E}$. This is fine too.

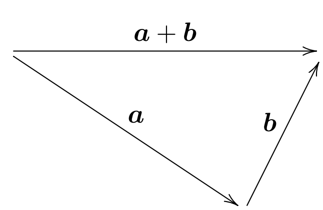
A vector can be represented as an arrow in space:



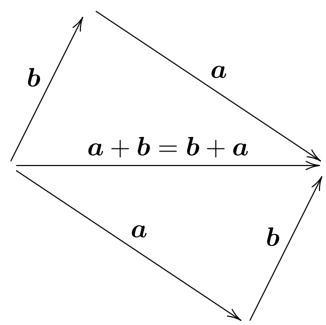
Given points A and B in space, we write $\overrightarrow{AB}$ to represent the vector going from the point A to the point B, as shown:



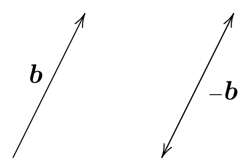
The sum of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is the vector obtained by going along one arrow, then the other.



Note that $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$:



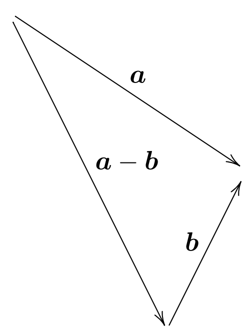
We define $-\boldsymbol{b}$ to be the vector obtained by going in the opposite direction to $\boldsymbol{b}$:



We write

$$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$$

Note that $(\boldsymbol{a} - \boldsymbol{b}) + \boldsymbol{b} = \boldsymbol{a}$:

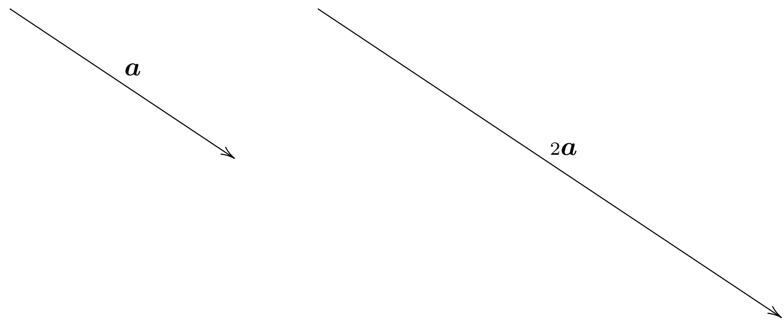


The *zero vector*, $\mathbf{0}$, goes nowhere; it is an arrow with no length, best seen as a single point. For any vector $\boldsymbol{a}$, we have

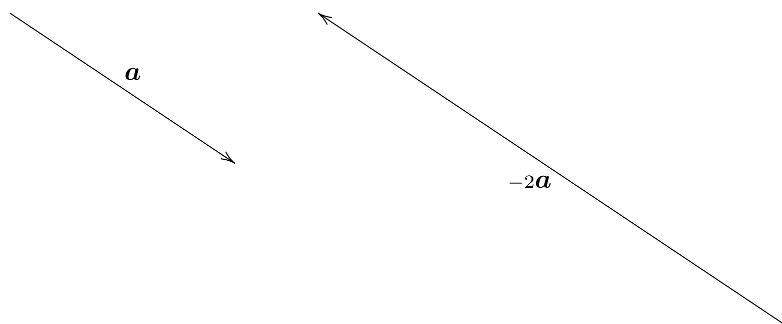
$$\boldsymbol{a} - \boldsymbol{a} = \mathbf{0}.$$

It is also convenient to be able to multiply a vector by a scalar. If $\boldsymbol{a}$ is a vector, and α is a positive scalar, then $\alpha\boldsymbol{a}$ is a vector in the same direction as

$\boldsymbol{a}$, but where we change its magnitude by α.



If $\alpha < 0$, we have to reverse the direction:



Finally, if $\alpha = 0$, then $\alpha\boldsymbol{a} = \mathbf{0}$.

1.2 Components and Magnitude

A vector, $\boldsymbol{a}$, in three-dimensional space has three scalar components, one in each of the x-, y-, and z- directions. If these three components are a_1, a_2 and a_3, then we write

$$\boldsymbol{a} = (a_1, a_2, a_3)$$

If we have another vector

$$\boldsymbol{b} = (b_1, b_2, b_3)$$

then

$$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

and

$$-\boldsymbol{b} = (-b_1, -b_2, -b_3)$$

Multiplication by a scalar is similar; if α is a scalar, then

$$\alpha\boldsymbol{a} = (\alpha a_1, \alpha a_2, \alpha a_3).$$

If we have points A and B, with coordinates (a_1, a_2, a_3) and (b_1, b_2, b_3) respectively, then

$$\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3).$$

In particular, if $0 = (0, 0, 0)$, and $A = (a_1, a_2, a_3)$ then $\overrightarrow{OA} = (a_1, a_2, a_3)$. We call the vector $\overrightarrow{OA}$ the *position vector* of A. Using position vectors blurs the line slightly between vectors and coordinates, but if we are careful this can be a useful thing to do.

An alternative way of writing vectors is sometimes useful in calculations. Let

$$\begin{aligned} \boldsymbol{i} &= (1, 0, 0) \\ \boldsymbol{j} &= (0, 1, 0) \\ \boldsymbol{k} &= (0, 0, 1) \end{aligned}$$

These are the unit vectors in the x- , y- and z- directions. Let $\boldsymbol{a} = (a_1, a_2, a_3)$. Then

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}.$$

To check this, note that

$$\begin{aligned} a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_2\boldsymbol{k} &= a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(0, 0, 1) \\ &= (a_1, 0, 0) + (0, a_2, 0) + (0, 0, a_3) \\ &= (a_1, a_2, a_3) \end{aligned}$$

Two vectors $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ are equal precisely when all of there components are equal, that is to say $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.

Definition 1.1 Let $\boldsymbol{a} = (a_1, a_2, a_3)$. Then the *magnitude* of $\boldsymbol{a}$ is given by the formula

$$|\boldsymbol{a}| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}.$$

The magnitude of a vector is its length. The magnitude $|\overrightarrow{AB}|$ is the distance from the point A to the point B. The above formula comes from working it out using Pythagoras' theorem.

Example 1.2 Calculate the distance from the point $A = (2, 1, 0)$ to the point $B = (-1, 3, 2)$.

Solution: Observe

$$\overrightarrow{BA} = (3, -2, 2)$$

So the distance between A and B is

$$|\overrightarrow{BA}| = \sqrt{3^2 + 2^2 + 2^2} = \sqrt{17}.$$

□

Definition 1.3 A vector $\boldsymbol{a}$ is termed a *unit vector* if its magnitude is one, that is $|\boldsymbol{a}| = 1$. For any vector $\boldsymbol{a} \neq \boldsymbol{0}$, we write $\hat{\boldsymbol{a}}$ to denote the unit vector with the same direction as $\boldsymbol{a}$.

Note that

$$\boldsymbol{a} = |\boldsymbol{a}|\hat{\boldsymbol{a}}$$

or equivalently

$$\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$

Example 1.4 Let

$$\boldsymbol{a} = (2, 3, 6)$$

The $\boldsymbol{a}$ has magnitude

$$|\boldsymbol{a}| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$

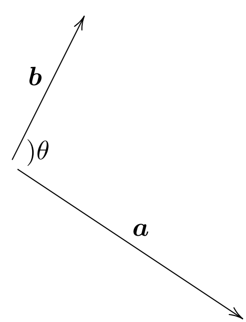
and we unit vector in the same direction

$$v(\hat{a}) = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a} = \frac{1}{7}(2, 3, 6) = \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right).$$

1.3 The Scalar Product

Definition 1.5 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be vectors, and let θ be the angle between them, as illustrated below. Then we define the *scalar product* of $\boldsymbol{a}$ and $\boldsymbol{b}$ by writing

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta.$$



The scalar product has the following properties:

- The scalar product of two vectors is a scalar, ie: just a number.
- For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we have $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$.
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, and head in the same direction, then $\theta = 0$, so $\cos\theta = 1$, and $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$.

- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, then $\theta = \frac{\pi}{2}$, so $\cos\theta = 0$, and $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. The converse is also true; if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, then, if they are not zero, $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.
- If $\boldsymbol{e}$ is a unit vector, then $\boldsymbol{a} \cdot \boldsymbol{e}$ is the component of $\boldsymbol{a}$ which lies in the direction of the vector $\boldsymbol{e}$.
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.
- For any three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, and scalars β and γ, we have

$$\boldsymbol{a} \cdot (\beta \boldsymbol{b} + \gamma \boldsymbol{c}) = \beta \boldsymbol{a} \cdot \boldsymbol{b} + \gamma \boldsymbol{a} \cdot \boldsymbol{c}.$$

Example 1.6 Let $\boldsymbol{F}$ be a force, and let $\boldsymbol{e}$ be a unit vector. Then the resultant force in the direction $\boldsymbol{e}$ is $\boldsymbol{F} \cdot \boldsymbol{e}$.

The following formula is used to compute the scalar product in most examples.

Proposition 1.7 *Let* $\boldsymbol{a} = (a_1, a_2, a_3)$ *and* $\boldsymbol{b} = (b_1, b_2, b_3)$. *Then*

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Proof: Write

$$\boldsymbol{a} = a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k} \qquad \boldsymbol{b} = b_1 \boldsymbol{i} + b_2 \boldsymbol{j} + b_3 \boldsymbol{k}.$$

Then by the last of the above properties

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= (a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k}) \cdot (b_1 \boldsymbol{i} + b_2 \boldsymbol{j} + b_3 \boldsymbol{k}) \\ &= a_1 b_1 \boldsymbol{i} \cdot \boldsymbol{i} + a_1 b_2 \boldsymbol{i} \cdot \boldsymbol{j} + \cdots + a_3 b_3 \boldsymbol{k} \cdot \boldsymbol{k} \end{aligned}$$

Now, the vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the unit vectors which lie in the directions of the x- , y- and z-axes. Thus they each have magnitude 1, and are perpendicular to each-other. By the above

$$\boldsymbol{i} \cdot \boldsymbol{i} = 1 \quad \boldsymbol{j} \cdot \boldsymbol{j} = 1 \quad \boldsymbol{k} \cdot \boldsymbol{k} = 1$$

and

$$\boldsymbol{i} \cdot \boldsymbol{j} = 0 \quad \boldsymbol{j} \cdot \boldsymbol{k} = 0 \quad \boldsymbol{k} \cdot \boldsymbol{i} = 0.$$

Hence, in the above sum

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 \boldsymbol{i} \cdot \boldsymbol{i} + a_1 b_2 \boldsymbol{i} \cdot vj + \cdots + a_3 b_3 \boldsymbol{k} \cdot \boldsymbol{k} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

as desired. □

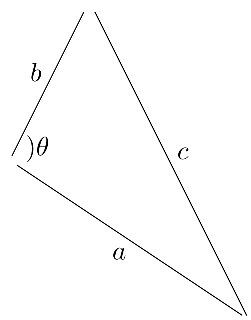
Example 1.8 Show that the vectors $\boldsymbol{a} = (3, 4, -9)$ and $\boldsymbol{b} = (3, 0, 1)$ are perpendicular.

Solution: We have

$$\boldsymbol{a} \cdot \boldsymbol{b} = 3 \times 3 + 4 \times 0 + (-9) \times 1 = 0$$

Hence $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular. □

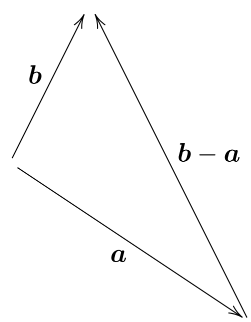
Proposition 1.9 *Consider a triangle, with sides and angles as shown:*



Then

$$a^2 + b^2 = c^2 + 2ab\cos\theta.$$

Proof: Consider vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as shown:



where $|\boldsymbol{a}| = a$, $|\boldsymbol{b}| = b$ and $\boldsymbol{b} - \boldsymbol{a} = c$.

Then

$$\begin{aligned} c^2 &= |\boldsymbol{b} - \boldsymbol{a}|^2 \\ &= (\boldsymbol{b} - \boldsymbol{a}) \cdot (\boldsymbol{b} - \boldsymbol{a}) \\ &= \boldsymbol{b} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{a} - 2\boldsymbol{a} \cdot \boldsymbol{b} \\ &= b^2 + a^2 - 2ab\cos\theta \end{aligned}$$

Rearranging, we see that

$$a^2 + b^2 = c^2 + 2ab\cos\theta$$

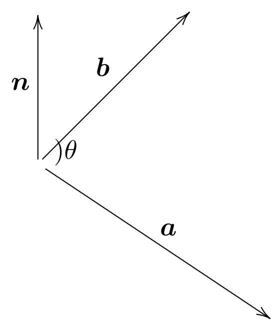
as required. □

1.4 The Vector Product

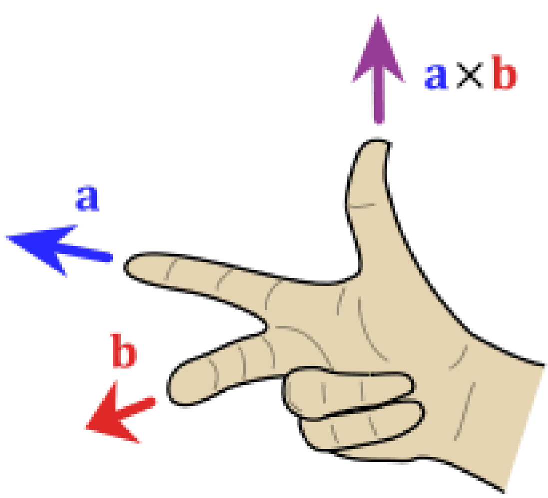
Definition 1.10 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be vectors in three-dimensional space, and let θ be the angle between them, as illustrated below. Then we define the *vectos product* of $\boldsymbol{a}$ and $\boldsymbol{b}$ by writing

$$\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta \boldsymbol{n}$$

where $\boldsymbol{n}$ is the unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ in the direction indicated.



To remember the direction of the vector $\boldsymbol{a} \times \boldsymbol{b}$ in the cross product, we use the following picture.



The vector product has the following properties:

- The scalar product of two vectors is another vector
- For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{a} \times \boldsymbol{b} = -vb \times va$. So order matters!

- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, meaning they head in the same direction, then $\theta = 0$, so $\sin\theta = 0$, and $\boldsymbol{a} \times \boldsymbol{b} = 0$.
- For any three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, and scalars β and γ, we have

$$\boldsymbol{a} \times (\beta\boldsymbol{b} + \gamma\boldsymbol{c}) = \beta\boldsymbol{a} \times \boldsymbol{b} + \gamma\boldsymbol{a} \times \boldsymbol{c}.$$

The following formula is what we use in practice to work out vector products. The proof is similar to that of the formula for scalar products.

Proposition 1.11 *Let* $\boldsymbol{a} = (a_1, a_2, a_3)$ *and* $\boldsymbol{b} = (b_1, b_2, b_3)$. *Then* $v a \times \boldsymbol{b}$ *is given by the* 3×3 *determinant*

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example 1.12 Let $\boldsymbol{a} = (1, 2, 3)$ and $\boldsymbol{b} = (0, 1, 1)$. Then

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix}$$

that is

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= (2 \times 1 - 3 \times 1)\boldsymbol{i} - (1 \times 1 - 3 \times 0)\boldsymbol{j} + (1 \times 1 - 2 \times 0)\boldsymbol{k} \\ &= -\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k} \\ &= (-1, -1, 1) \end{aligned}$$

Definition 1.13 The *triple scalar product* of vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is the scalar $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$.

If $\boldsymbol{a} = (a_1, a_2, a_3)$, $\boldsymbol{b} = (b_1, b_2, b_3)$ and $\boldsymbol{c} = (c_1, c_2, c_3)$ then we can write the triple product as a determinant

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The absolute value of the scalar triple product is the volume of a parallelopiped, with edges given by vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. It has the following properties, for vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

- $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b})$
- $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = -\boldsymbol{a} \cdot (\boldsymbol{c} \times \boldsymbol{b}) = -\boldsymbol{b} \cdot (\boldsymbol{a} \cdot \boldsymbol{c})$
- If any two of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are parallel, then $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = 0$.

Example 1.14 Let $\boldsymbol{a} = (1, -2, 3)$, $\boldsymbol{b} = (1, 1, 1)$ and $\boldsymbol{c} = (1, 0, 1)$. Then

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} 1 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

that is

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = 1 \times (1 \times 1 - 1 \times 0) - (-2) \times (1 \times 1 - 1 \times 1) + 3 \times (1 \times 1 - 1 \times 1) = 1.$$

As well as a triple scalar product, there is a triple vector product.

Definition 1.15 The *triple vector product* of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is the vector $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$.

As a warning, the brackets matter here:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \neq (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}.$$

Proposition 1.16 *Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be vectors in three-dimensional space. Then*

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}.$$

The proof of this result is simply a case of using the formulae to work both the left and right-hand side out in terms of components and showing they are equal. This is tedious. However, let us do an example.

Example 1.17 Let

$$\boldsymbol{a} = (2, -2, 3) \qquad \boldsymbol{b} = (1, -3, 4) \qquad \boldsymbol{c} = (0, -3, 2)$$

Then

$$\boldsymbol{b} \times \boldsymbol{c} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -3 & 4 \\ 0 & -3 & 2 \end{vmatrix} = (6, -2, -3)$$

so

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -2 & 3 \\ 6 & -2 & 3 \end{vmatrix} = (12, 24, 8).$$

On the other hand

$$\boldsymbol{a} \cdot \boldsymbol{c} = 12 \qquad \boldsymbol{a} \cdot \boldsymbol{b} = 20$$

and

$$(\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} = 12(1, -3, 4) - 20(0, -3, 2) = (12, 24, 8).$$

The formula for the triple product can be used to prove the following, which is sometimes called the *Jacobi identity*

Proposition 1.18

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{0}.$$

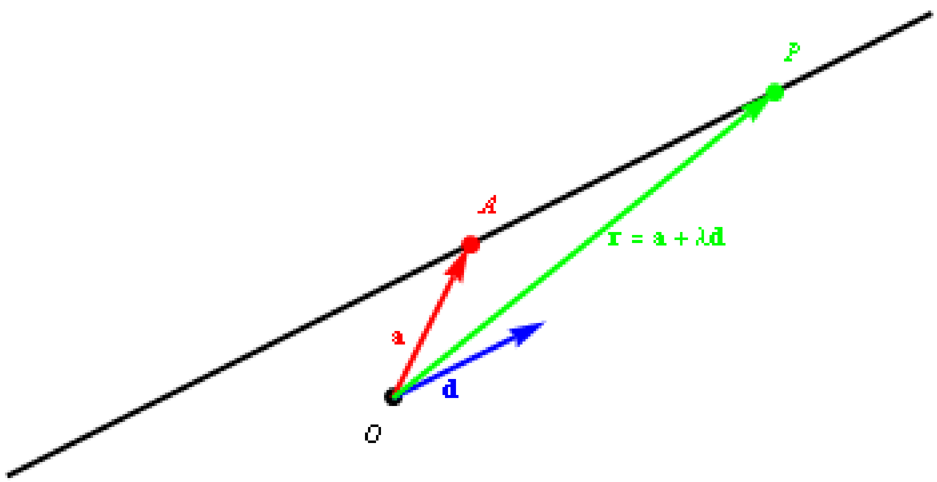
□

1.5 Lines

Let L be a line in three-dimensional space. Let A be a fixed point on L, and let $\boldsymbol{d}$ be a vector in the direction of L. Let A have position vector $\overrightarrow{OA} = \boldsymbol{a}$. Then a point P on L has position vector

$$\overrightarrow{OP} = \boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{d}$$

where t is a varying scalar. We call the above the *vector equation* of the line L



When formulating this equation, it doesn't matter which point A and which direction vector $\boldsymbol{d}$ we choose; any point on the line and any vector parallel to the line will do.

If we have two position vectors, $\boldsymbol{a}$ and vb for points on the line, then the vector $\boldsymbol{b} - va$ is parallel to the line, so the line has equation

$$\boldsymbol{r} = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a}).$$

In components, let

$$\boldsymbol{r} = (r_1, r_2, r_3) \qquad \boldsymbol{a} = (a_1, a_2, a_3) \qquad \boldsymbol{b} = (b_2, b_2, b_3)$$

Then

$$r_1 = a_1 + t(b_1 - a_1) \qquad r_2 = a_2 + t(b_2 - a_2) \qquad r_3 = a_3 + t(b_3 - a_3)$$

Rearranging:

$$t = \frac{r_1 - a_1}{b_1 - a_1} \qquad t = \frac{r_2 - a_2}{b_2 - a_2} \qquad t = \frac{r_3 - a_3}{b_3 - a_3}$$

So the line is described by the equations

$$\frac{r_1 - a_1}{b_1 - a_1} = \frac{r_2 - a_2}{b_2 - a_2} = \frac{r_3 - a_3}{b_3 - a_3}$$

A point (r_1, r_2, r_3) satisfying these equations is on the line. However, the earlier vector equation is usually more useful.

Now consider a point C and a line L. How do we work out the shortest distance from C to L?

Well, for a point P on L, we calculate the magnitude of the vector $\overrightarrow{CP}$, and then use calculus to work out the minimum.

Because of the formulae involved, it turns out to be easier to find the minimum value of $|\overrightarrow{CP}|^2$, and then work out the square root.

Example 1.19 Find the distance from the point $(1, 0, 1)$ to the line which goes through the points $(1, 1, 0)$ and $(0, -1, -1)$.

Solution: Let $C = (1, 0, 1)$. Let P be a point on the line. Let $D = |\overrightarrow{CP}|$. Then P has position vector

$$\boldsymbol{r} = (1, 1, 0) + t((0, -1, -1) - (1, 1, 0) = (1 - t, 1 - 2t, t)$$

Then

$$\overrightarrow{CP} = (1 - t, 1 - 2t, t) - (1, 0, 1) = (-t, 1 - 2t, -1 - t)$$

and

$$D^2 = |\overrightarrow{CP}|^2 = t^2 + (1 - 2t)^2 + (t - 1)^2 = 6t^2 - 6t + 2.$$

We want to work out the minimum of D^2. This will occur when

$$\frac{d(D^2)}{dt} = 0.$$

Observe

$$\frac{d(D^2)}{dt} = 12t - 6$$

so $\frac{d(D^2)}{dt} = 0$ means that $12t - 6 = 0$, that is

$$t = \frac{1}{2}.$$

Feeding this back into the formula for D^2, we see that our minimum distance, and so the distance from the point to the line is given by

$$D^2 = 6\left(\frac{1}{2}\right)^2 - 6\left(\frac{1}{2}\right) + 2 = \frac{1}{2}$$

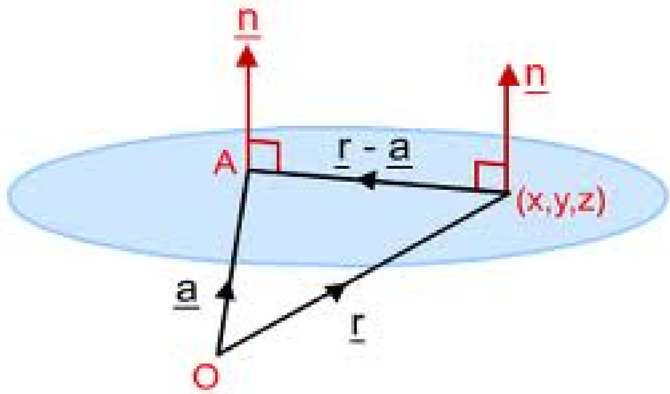
and so our distance is

$$D = \frac{1}{\sqrt{2}}.$$

□

1.6 Planes

Consider a plane, Π, in three dimensional space, with normal vector $\boldsymbol{n}$. Let A be a fixed point on the place with position vector $\boldsymbol{a}$, and let P be general point on the plane with position vector $\boldsymbol{r}$, as shown.



Then $\overrightarrow{AR} = \boldsymbol{r} - \boldsymbol{a}$ is a vector within the plane, so it is perpendicular to the normal vector $\boldsymbol{n}$. Hence $(\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{n} = 0$.

In other words, the plane Π has equation

$$\boldsymbol{r} \cdot \boldsymbol{n} = c$$

where c is the constant, $c = \boldsymbol{a} \cdot \boldsymbol{n}$.

To find the normal vector $\boldsymbol{n}$, let A, B and C be three points in the plane. Then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are vectors within the plane. By definition of the vector product,

$$\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{AC}$$

will be perpendicular to Π, as long as it isn't zero.

Example 1.20 Let $A = (1, 2, 0)$, $B = (1, 0, 1)$ and $C = (0, 1, 2)$. Find the equation of the plane through A, B and C.

Solution: Observe

$$\overrightarrow{AB} = (1, 0, 1) - (1, 2, 0) = (0, -2, 1)$$

and

$$\overrightarrow{AC} = (0, 1, 2) - (1, 2, 0) = (-1, -1, 2)$$

so

$$\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & -2 & 1 \\ -1 & -1 & 2 \end{vmatrix} = (-3, -1, -2)$$

We have $\boldsymbol{a} = (1, 2, 0)$ so

$$c = \boldsymbol{a} \cdot \boldsymbol{n} = (1, 2, 0) \cdot (-3, -1, -2) = -5$$

Thus the plane has equation

$$\boldsymbol{r} \cdot (-3, -1, -2) = -5$$

In components $\boldsymbol{r} = (r_1, r_2, r_3)$, this is

$$-3r_1 - r_2 - 2r_3 = -5$$

or, rearranging

$$3r_1 + r_2 + 2r_3 = 5.$$

□

In general, a plane has an equation of the form

$$\alpha r_1 + \beta r_2 + \gamma r_3 = c$$

where α, β, γ and c are constants.

Proposition 1.21 *Let* Π *be a plane, and let* $\boldsymbol{n}$ *be a* unit *normal vector to* Π. *Let* C *be a point. Then the distance from* C *to* Π *is given by*

$$D = |\overrightarrow{AC} \cdot \boldsymbol{n}|$$

where A *is any point in the plane.*

Proof: By definition of the scalar product, the component of $\overrightarrow{AC}$ that is perpendicular to Π is given by $\overrightarrow{AC} \cdot \boldsymbol{n}$; we take the absolute value to eliminate any possible minus sign. Drawing a picture tells us that this is the distance we need. □

Note that in the above formula, we need $\boldsymbol{n}$ to be a *unit* normal vector. If $\boldsymbol{n}$ is not a unit vector (and it won't be if we work it out as before) we replace it by the unit vector in the same direction, $\boldsymbol{u} = \frac{\boldsymbol{n}}{|\boldsymbol{n}|}$. It does not matter which point A in the plane we pick.

Example 1.22 Find the distance from the point $C = (1, 1, 1)$ to the plane Π in example 1.20.

Solution: As in example 1.20, we have normal vector

$$\boldsymbol{n} = (-3, -1, -2)$$

This is not a unit vector; we have $|\boldsymbol{n}|^2 = 3^2 + 1^2 + 2^2 = 14$. So we replace with a unit normal vector

$$\boldsymbol{u} = \frac{\boldsymbol{n}}{|\boldsymbol{n}|} = \left(-\frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}\right)$$

The point A in the plane has position vector $(1, 2, 0)$. So the distance we require is

$$|\boldsymbol{u} \cdot \boldsymbol{a}| = \frac{1}{\sqrt{14}}|-3 \times 1 - 1 \times 2 - 2 \times 0| = \frac{5}{\sqrt{14}}.$$

□

2 Differentiation and Vectors

2.1 Differentiation of Vector-valued Functions

An ordinary function, with values a scalar is called a *scalar-valued function.* A function whose values are a vector is called a*vector-valued function.* If$\boldsymbol{v}(t)$ is a vector-valued function, we can write it

$$\boldsymbol{v}(t) = (v_1(t), v_2(t), v_3(t))$$

where $v_1(t)$, $v_2(t)$ and $v_3(t)$ are scalar-valued functions.

Definition 2.1 We define the *derivative* of $\boldsymbol{v}(t)$:

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{v}'(t) = (v_1'(t), v_2'(t), v_3'(t)).$$

The derivative of a vector valued function has the following properties:

- The derivative $\boldsymbol{v}'(t)$ of $\boldsymbol{v}(t)$ is another vector-valued function.
- $\frac{d\boldsymbol{c}}{dt} = 0$ is $\boldsymbol{c}$ is a constant vector. Conversely, if $\frac{d\boldsymbol{v}}{dt} = 0$, then $\boldsymbol{v}(t) = \boldsymbol{c}$ for some constant vector $\boldsymbol{c}$.
- If C is a constant scalar, then
$$\frac{d}{dt}(C\boldsymbol{v}) = C\boldsymbol{v}'(t)$$
- If $\alpha(t)$ is a scalar-valued function, then
$$\frac{d}{dt}(\alpha\boldsymbol{v}) = \frac{d\alpha}{dt}\boldsymbol{v} + \alpha\frac{d\boldsymbol{v}}{dt}.$$
- If $\boldsymbol{v}(t)$ and $\boldsymbol{w}(t)$ are vector-valued functions, then
$$\frac{d}{dt}(\boldsymbol{v} + \boldsymbol{w}) = \frac{d\boldsymbol{v}}{dt} + \frac{d\boldsymbol{w}}{dt}$$
- $$\frac{d}{dt}(\boldsymbol{v} \cdot \boldsymbol{w}) = \frac{d\boldsymbol{v}}{dt} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \frac{d\boldsymbol{w}}{dt}$$
- $$\frac{d}{dt}(\boldsymbol{v} \times \boldsymbol{w}) = \frac{d\boldsymbol{v}}{dt} \times \boldsymbol{w} + \boldsymbol{v} \times \frac{d\boldsymbol{w}}{dt}$$

We can prove these formulae by splitting the vector into components. However, if we can, it's usually better not to split into components to manipulate derivatives of vector-valued functions, though it's sometimes necessary.

Proposition 2.2 *Let $\boldsymbol{v}(t)$ be a vector-valued function such that $|\boldsymbol{v}(t)|$ is constant. Then $\boldsymbol{v}(t)$ and $\boldsymbol{v}'(t)$ are perpendicular.*

Proof: We have that

$$|\boldsymbol{v}(t)|^2 = \boldsymbol{v}(t) \cdot \boldsymbol{v}(t)$$

is constant. Hence

$$\frac{d}{dt}(\boldsymbol{v}(t) \cdot \boldsymbol{v}(t)) = 0$$

ie:

$$\boldsymbol{v}(t) \cdot \boldsymbol{v}'(t) + \boldsymbol{v}'(t) \cdot \boldsymbol{v}(t) = 0$$

thiat is to say $2\boldsymbol{v}(t) \cdot \boldsymbol{v}'(t) = 0$.

Thus $\boldsymbol{v}(t) \cdot \boldsymbol{v}'(t) = 0$ and we are done. □

Example 2.3 Let

$$\boldsymbol{r}(t) = \cos(\omega t)\boldsymbol{i} + \sin(\omega t)\boldsymbol{j}$$

Then

$$|\boldsymbol{r}(t)|^2 = \cos^2(\omega t) + \sin^2(\omega t) = 1$$

so $\boldsymbol{r}(t)$ is constant.

Observe

$$\boldsymbol{r}'(t) = -\omega \sin(\omega t)\boldsymbol{i} + \omega \cos(\omega t)\boldsymbol{j}$$

and

$$\boldsymbol{r}(t) \cdot \boldsymbol{r}'(t) = -\omega \cos(\omega t) \sin(\omega t) + \omega \cos(\omega t) \sin(\omega t) = 0$$

Example 2.4 A particle has position $\boldsymbol{r}(t)$ at time t, and constant mass m. The *angular momentum* is defined to be the vector

$$\boldsymbol{P} = \boldsymbol{r} \times m\frac{d\boldsymbol{r}}{dt}$$

Show that

$$\frac{d\boldsymbol{P}}{dt} = \boldsymbol{r} \times \boldsymbol{F}$$

where $\boldsymbol{F} = m\frac{d^2\boldsymbol{r}}{dt^2}$.

Solution: We have

$$\frac{d\boldsymbol{P}}{dt} = \frac{d}{dt}\left(\boldsymbol{r} \times m\frac{d\boldsymbol{r}}{dt}\right) = \frac{d\boldsymbol{r}}{dt} \times m\frac{d\boldsymbol{r}}{dt} + \boldsymbol{r} \times m\frac{d^2\boldsymbol{r}}{dt^2}$$

Now, for any vector $\boldsymbol{a}$ we have $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$. So

$$\frac{d\boldsymbol{P}}{dt} = \boldsymbol{r} \times m\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{r} \times \boldsymbol{F}$$

as required. □

Now, a curve, C, in three-dimensional space can be defined parametrically by stating that the position vector of a point on the curve is given by a vector-valued function

$$\boldsymbol{r}(t) = (x(t), y(t), z(t))$$

Definition 2.5 The *tangent vector* to the curve C at the point $\boldsymbol{r}(t)$ is the vector $\boldsymbol{r}'(t)$. The *unit tangent* is the unit vector $\boldsymbol{t}(t)$ in the direction $\boldsymbol{r}'(t)$.

Example 2.6 We define the *helix* to be the curve given by the parametric equation

$$\boldsymbol{r}(t) = (a\cos(\omega t), a\sin(\omega t), ct)$$

where ω, a and c are constants. Find the tangent vector and unit tangent.

Solution: We have tangent vector

$$\frac{d\boldsymbol{r}}{dt} = (-a\omega\sin(\omega t), a\omega\cos(\omega t), c)$$

Observe

$$\left|\frac{d\boldsymbol{r}}{dt}\right| = \sqrt{a^2\omega^2\sin^2(\omega t) + a^2\omega^2\cos^2(\omega t) + c^2} = \sqrt{a^2\omega^2 + c^2}$$

so our unit tangent is

$$\boldsymbol{t}(t) = \frac{1}{\sqrt{a^2\omega^2 + c^2}}(-a\omega\sin(\omega t), a\omega\cos(\omega t), c)$$

□

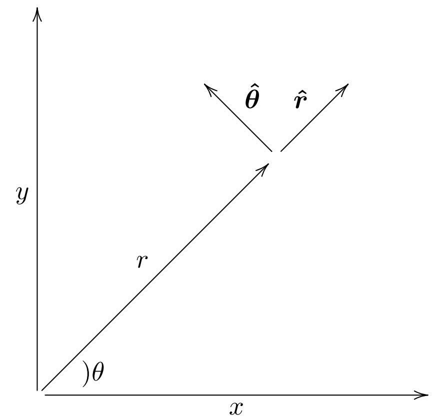
2.2 Plane polar Coordinates

Ordinary Cartesian coordinates are not always the best choice for studying a physical or geometric problem, especially when there is some sort of symmetry. Plane polar coordinates are useful for many two-dimensional problems. We look at three-dimensional generalisations later on.

Consider a point in the plane with position vector

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$$

Let $r = |\boldsymbol{r}|$, and let θ be the angle between the position vector and the x-axis, as in the following diagram.



Then

$$x = r\cos\theta \qquad y = r\sin\theta$$

Let $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ be unit vectors in the direction of increasing r and θ, as shown. Note that for a function $\boldsymbol{r}(t)$, in general the unit vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ change with t, ie: they are vector-valued functions $\hat{\boldsymbol{r}}(t)$ and $\hat{\boldsymbol{\theta}}(t)$.

When working in plane polar coordinates, we express vectors with components in the directions of $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ rather than $\boldsymbol{i}$ and $\boldsymbol{j}$. However, we need to differentiate $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$.

Looking at the picture again, we have the following.

Proposition 2.7

$$\hat{\boldsymbol{r}}\cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j} \qquad \hat{\boldsymbol{\theta}} = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}$$

□

Notice that $\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\theta}} = 0$.

Proposition 2.8 *Let $\boldsymbol{r}(t)$ be a vector-valued function. Then*

$$\frac{d\hat{\boldsymbol{r}}}{dt} = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}} \qquad fracd\hat{\boldsymbol{\theta}}dt = -\frac{d\theta}{dt}\hat{\boldsymbol{r}}$$

Proof: We have, by the chain rule for derivatives

$$\frac{d\hat{\boldsymbol{r}}}{dt} = \frac{d}{dt}(\cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}) = (-\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j})\frac{d\theta}{dt} = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}}$$

The second formula arises similarly. □

If $\boldsymbol{r}(t)$ is the position vector of a particle, then it has velocity $\frac{d\boldsymbol{r}(t)}{dt}$ and acceleration $\frac{d^2\boldsymbol{r}(t)}{dt^2}$. We would like to express these in terms of $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$.

Firstly, note that

$$\boldsymbol{r} = r\hat{\boldsymbol{r}}$$

Proposition 2.9

$$\frac{d\boldsymbol{r}}{dt} = \frac{dr}{dt}\hat{\boldsymbol{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}$$

and

$$\frac{d^2\boldsymbol{r}}{dt^2} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{\boldsymbol{r}} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\hat{\boldsymbol{\theta}} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{\boldsymbol{r}} + \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\hat{\boldsymbol{\theta}}$$

Proof: Since $\boldsymbol{r} = r\hat{\boldsymbol{r}}$ we have, by the previous proposition

$$\frac{d\boldsymbol{r}}{dt} = \frac{dr}{dt}\hat{\boldsymbol{r}} + r\frac{d\hat{\boldsymbol{r}}}{dt} = \frac{dr}{dt}\hat{\boldsymbol{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}$$

which is the first formula.

Now

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{d^2r}{dt^2}\hat{\boldsymbol{r}} + \frac{dr}{dt}\frac{d\hat{\boldsymbol{r}}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\frac{d^\theta}{dt^2} + r\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{dt}$$

Gathering terms and using the above proposition

$$\frac{d^2\boldsymbol{r}}{dt^2} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{\boldsymbol{r}} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\hat{\boldsymbol{\theta}}$$

which is the second formula.

Finally note that

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{1}{r}\left(2r\frac{dr}{dt}\frac{d\theta}{dt} + r^2\frac{d^2\theta}{dt^2}\right) = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

and we have the final formula. □

If we have a particle, with position vector $\boldsymbol{r}(t)$, mass m, being acted on by a force

$$\boldsymbol{F} = F_r\hat{\boldsymbol{r}} + F_\theta\hat{\boldsymbol{\theta}}$$

then Newton's second law says

$$\boldsymbol{F} = m\frac{d^2\boldsymbol{r}}{dt^2}$$

so, by the above

$$F_r = m\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) \qquad F_\theta = \frac{m}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

Example 2.10 For a planet, with mass m, orbiting the sun (which has mass M), we have

$$F_r = -\frac{GMm}{r^2} \qquad F_\theta = 0$$

where G is the universal gravitational constant.

Hence, by the second of the above formulae,

$$\frac{m}{r}\left(r^2\frac{d\theta}{dt}\right) = 0$$

It follows that the value

$$h = r^2\frac{d\theta}{dt}$$

is constant.

Example 2.11 If a particle moves in a circle of radius R, then $r = R$ (a constant), and

$$\frac{dr}{dt} = 0 \qquad \frac{d^2r}{dt^2} = 0$$

Thus

$$\frac{d\boldsymbol{r}}{dt} = R\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} \qquad \frac{d^2\boldsymbol{r}}{dt^2} = -R\left(\frac{d\theta}{dt}\right)\hat{\boldsymbol{r}} + R\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}}$$

Let

$$v = \left|\frac{d\boldsymbol{r}}{dt}\right|$$

Then by the first of the above formulae

$$v = R\frac{d\theta}{dt}$$

so

$$\frac{d\theta}{dt} = \frac{v}{R} \qquad \frac{d^\theta}{dt^2} = \frac{1}{R}\frac{dv}{dt}$$

and the second formula becomes

$$\frac{d^2\boldsymbol{r}}{dt^2} = -\frac{v^2}{R}\hat{\boldsymbol{r}} + \frac{dv}{dt}\hat{\boldsymbol{\theta}}$$

2.3 Partial Derivatives

A scalar-valued function $\phi(\boldsymbol{r})$ of a vector $\boldsymbol{r}$ in three-dimensional space is sometimes called a *scalar field.* If $\boldsymbol{r} = (x, y, z)$, we can view ϕ as a function of three variables $\phi(x, y, z)$.

Definition 2.12 The *partial derivative*

$$\frac{\partial\phi}{\partial x}$$

is what we get if we differentiate ϕ with respect to x while treating the other variables y and z as constants.

The partial derivatives

$$\frac{\partial\phi}{\partial y} \qquad \frac{\partial\phi}{\partial z}$$

are defined similarly.

This is best seen with an example.

Example 2.13 Let

$$\phi(x, y, z) = x^2 + y^2 + z^2 + xye^z$$

Then

$$\frac{\partial \phi}{\partial x} = 2x + ye^z$$

(treating y and z as constant)

$$\frac{\partial \phi}{\partial y} = 2y + xe^z$$

(treating x and z as constant)

$$\frac{\partial \phi}{\partial z} = 2z + xye^z$$

(treating x and y as constant).

We can also form a variety of second partial derivatives. For example

$$\frac{\partial^2 \phi}{\partial x^2}$$

means differentiate with respect to x twice, and

$$\frac{\partial^2 \phi}{\partial y \partial x}$$

means differentiate with respect to x then y.

Actually, the order of differentiateion does not really matter as the following theorem (which we will not prove) shows.

Theorem 2.14 *Let ϕ be a scalar field where all partial derivatives, second partial derivatives, and so on exist. Then*

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$$

and similarly in the other variables. □

We call a scalar field where all partial derivatives of all orders exist, as required for the above *smooth* or *infinitely differentiable*

Let us verify the above in an example.

Example 2.15 Let

$$\phi(x, y, z) = x^2 + y^2 + z^2 + xye^z$$

as in the above example.

Then

$$\frac{\partial^2\phi}{\partial x\partial y} = e^z \qquad \frac{\partial^2\phi}{\partial y\partial x} = e^z$$

Also

$$\frac{\partial^2\phi}{\partial y\partial z} = xe^z \qquad \frac{\partial^2\phi}{\partial z\partial y} = xe^z$$

and

$$\frac{\partial^2\phi}{\partial x\partial z} = ye^z \qquad \frac{\partial^2\phi}{\partial z\partial x} = ye^z$$

The following result, which we will not prove here, is sometimes called the *chain rule* for partial derivatives.

Theorem 2.16 *Let $\phi(\boldsymbol{r})$ be a scalar field, where as usual $\boldsymbol{r} = (x, y, z)$. Let $\boldsymbol{v}(t)$ be a vector-valued function. Write $\boldsymbol{v}(t) = (v_1(t), v_2(t), v_3(t))$. Then*

$$\frac{d\phi(\boldsymbol{v}(t))}{dt} = \frac{\partial\phi}{\partial x}\frac{dv_1}{dt} + \frac{\partial\phi}{\partial y}\frac{dv_2}{dt} + \frac{\partial\phi}{\partial z}\frac{dv_3}{dt}.$$

□

We will return to the chain rule for partial derivatives later on.

2.4 Vector Fields

A vector-valued function $\boldsymbol{V}(\boldsymbol{r})$ of a vector $\boldsymbol{r}$ in three-dimensional space is called a *vector field*. If $\boldsymbol{r} = (x, y, z)$, we can view $\boldsymbol{V}$ as a vector-valued function of three variables, $\boldsymbol{V}(x, y, z)$.

We can write a vector-field $\boldsymbol{V}(x, y, z)$ in terms of its components

$$\boldsymbol{V}(x, y, z) = V_1(x, y, z)\boldsymbol{i} + V_2(x, y, z)\boldsymbol{j} + V_3(x, y, z)\boldsymbol{k}$$

Example 2.17 The function

$$\boldsymbol{V}(\boldsymbol{r}) = |\boldsymbol{r}|^2\boldsymbol{r}$$

is a vector field. In terms of components, if $\boldsymbol{r} = (x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, then $|\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2}$, and

$$\boldsymbol{V}(x, y, z) = (x^2 + y^2 + z^2)(x, y, z) = (x^2 + y^2 + z^2)(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}).$$

Viewed in terms of components, we define the partial derivatives of a vector field:

$$\frac{\partial\boldsymbol{V}}{\partial x} = \frac{\partial V_1}{\partial x}\boldsymbol{i} + \frac{\partial V_2}{\partial x}\boldsymbol{j} + \frac{\partial V_3}{\partial x}\boldsymbol{k}$$
$$\frac{\partial\boldsymbol{V}}{\partial x} = \frac{\partial V_1}{\partial x}\boldsymbol{i} + \frac{\partial V_2}{\partial x}\boldsymbol{j} + \frac{\partial V_3}{\partial x}\boldsymbol{k}$$

and

$$\frac{\partial \boldsymbol{V}}{\partial x} = \frac{\partial V_1}{\partial x}\boldsymbol{i} + \frac{\partial V_2}{\partial x}\boldsymbol{j} + \frac{\partial V_3}{\partial x}\boldsymbol{k}$$

As before, we have the formula

$$\frac{\partial^2 \boldsymbol{V}}{\partial x \partial y} = \frac{\partial^2 \boldsymbol{V}}{\partial y \partial x}$$

(and similarly for the other variables) provided all partial derivatives, second partial derivatives, and so on exist.

Example 2.18 For the above function

$$\boldsymbol{V}(\boldsymbol{r}) = |\boldsymbol{r}|^2\boldsymbol{r} = (x^2+y^2+z^2)(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})$$

we have, by the product rule for differentiation

$$\frac{\partial \boldsymbol{V}}{\partial x} = (x^2+y^2+z^2)\boldsymbol{i} + 2x(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}) = (3x^2+y^2+z^2)\boldsymbol{i} + 2xy\boldsymbol{j} + 2xz\boldsymbol{k}.$$

Similarly

$$\frac{\partial \boldsymbol{V}}{\partial y} = 2xy\boldsymbol{i} + (x^2+3y^2+z^2)\boldsymbol{j} + 2yz\boldsymbol{k}$$

and

$$\frac{\partial \boldsymbol{V}}{\partial z} = 2xz\boldsymbol{i} + 2yz\boldsymbol{j} + (x^2+y^2+3z^2)\boldsymbol{k}.$$

Now observe

$$\frac{\partial^2 \boldsymbol{V}}{\partial y \partial x} = \frac{\partial}{\partial y}\big((3x^2+y^2+z^2)\boldsymbol{i} + 2xy\boldsymbol{j} + 2xz\boldsymbol{k}\big) = 2y\boldsymbol{i} + 2x\boldsymbol{j}$$

and

$$\frac{\partial^2 \boldsymbol{V}}{\partial x \partial y} = \frac{\partial}{\partial x}\big(2xy\boldsymbol{i} + (x^2+3y^2+z^2)\boldsymbol{j} + 2yz\boldsymbol{k}\big) = 2y\boldsymbol{i} + 2x\boldsymbol{j}$$

3 Grad, Div and Curl

3.1 The Gradient of a Scalar Field

Definition 3.1 Let $\phi(\boldsymbol{r})$ be a scalar field, where as usual $\boldsymbol{r} = (x, y, z)$. Then we define the *gradient* of ϕ by writing

$$grad\,\phi = \frac{\partial \phi}{\partial x}\boldsymbol{i} + \frac{\partial \phi}{\partial y}\boldsymbol{j} + \frac{\partial \phi}{\partial z}\boldsymbol{k}) = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right).$$

We also write

$$grad\,\phi = \boldsymbol{\nabla}\phi$$

where $\boldsymbol{\nabla}$ is the vector operation

$$\boldsymbol{\nabla} = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$$

Note that the gradiant of a scalar field is a vector field.

Example 3.2 For a vector $\boldsymbol{r} = (x, y, z)$, let $r = |\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2}$. Consider the scalar field

$$U(\boldsymbol{r}) = \frac{1}{r}.$$

Observe

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$

so

$$\frac{\partial U}{\partial x} = \frac{dU}{dr}\frac{\partial r}{\partial x} = -\frac{x}{r^3}.$$

Alternatively, we could work this out directly. Similarly,

$$\frac{\partial U}{\partial y} = -\frac{y}{r^3} \qquad \frac{\partial U}{\partial z} = -\frac{z}{r^3}.$$

Hence

$$\boldsymbol{\nabla} U = \frac{\partial U}{\partial x}\boldsymbol{i} + \frac{\partial U}{\partial y}\boldsymbol{j} + \frac{\partial U}{\partial z}\boldsymbol{k} = \frac{-x\boldsymbol{i} - y\boldsymbol{j} - z\boldsymbol{k}}{r^3} = -\frac{\boldsymbol{r}}{r^3}.$$

Now, let $\phi(\boldsymbol{r})$ be a scalar field, and let $\boldsymbol{v}(t)$ be a vector-valued function. Write $\boldsymbol{v}(t) = (v_1(t), v_2(t), v_3(t))$. Recall that the *chain rule for partial derivatives* tells us that:

$$\frac{d\phi(\boldsymbol{v}(t))}{dt} = \frac{\partial \phi}{\partial x}\frac{dv_1}{dt} + \frac{\partial \phi}{\partial y}\frac{dv_2}{dt} + \frac{\partial \phi}{\partial z}\frac{dv_3}{dt}.$$

In other words, in terms of the gradient of ϕ, we can write the chain rule for partial derivatives in the form:

$$\frac{d\phi(\boldsymbol{v}(t))}{dt} = (\boldsymbol{\nabla}\phi) \cdot \frac{d\boldsymbol{v}}{dt}.$$

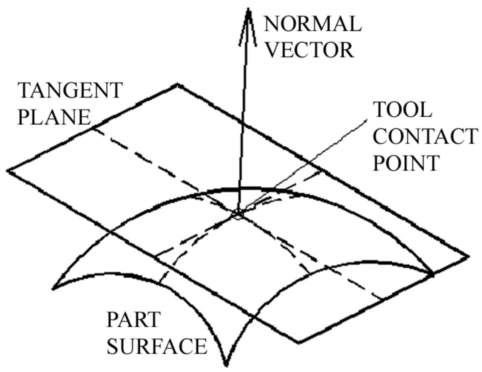
Definition 3.3 Let $\phi(x, y, z)$ be a scalar field, and let c be a constant. Then the set of points (x, y, z) satisfying the equation $\phi(x, y, z) = c$ is called a *level surface* for ϕ.

Example 3.4 Let $\phi(x, y, z) = x^2 + y^2 + z^2$. Let $R > 0$ be a constant. Then the level surface

$$\phi(x, y, z) = R^2$$

is a sphere with radius R.

Now, if (x_0, y_0, z_0) is a point on the level surface $\phi(x, y, z) = c$, then the *tangent plane* is the plane which just touches the level surface at (x_0, y_0, z_0) as shown.



Proposition 3.5 *Let $\phi(\boldsymbol{r})$ be a scalar field. Let $\phi(x_0, y_0, z_0) = c$, so that (x_0, y_0, z_0) belongs to the level surface $\phi(x, y, z) = c$. Then the vector $\boldsymbol{\nabla}\phi(x_0, y_0, z_0)$ is a normal vector to the tangent plane at (x_0, y_0, z_0) of the level surface.*

Proof: Let $\boldsymbol{v}(t)$ be a vector-valued function that takes values within the level surface $\phi(x, y, z) = c$. Then $\phi(\boldsymbol{v}(t)) = c$ for all t. Thus

$$\frac{d\phi(\boldsymbol{v}(t))}{dt} = 0$$

that is, by the chain rule

$$(\boldsymbol{\nabla}\phi) \cdot \frac{d\boldsymbol{v}}{dt} = 0$$

But geometrically, any vector within the tangent plane at (x_0, y_0, z_0) is given by $\frac{d\boldsymbol{v}}{dt}(t_0)$ for some function $\boldsymbol{v}(t)$ within the level surface, where $\boldsymbol{v}(t_0) = (x_0, y_0, z_0)$. Hence $v\nabla\phi(x_0, y_0, z_0)$ is perpendicular to all vectors within the tangent plane. This makes it a normal vector. □

Example 3.6 Given the surface $x^3y^2z = 12$, find the tangent plane at the point $(1, -2, 3)$.

Solution: Consider the function $\phi(x, y, z) = x^3y^2z$. Then we are looking at a level surface to ϕ. Observe

$$\boldsymbol{\nabla}\phi = 3x^2y^2z\boldsymbol{i} + 2x^3yz\boldsymbol{j} + x^3y^2\boldsymbol{k}$$

so at the point $(1, -2, 3)$ we have a normal vector to the tangent plane.

$$\boldsymbol{\nabla}\phi(1, -2, 3) = 36\boldsymbol{i} - 12\boldsymbol{j} + 4\boldsymbol{k}$$

Let us divide by 4 to get an easier to handle normal vector

$$\boldsymbol{n} = 9\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k} = (9, -3, 1).$$

Then the tangent plane has equation

$$(\boldsymbol{r} - (1, -2, 3)) \cdot \boldsymbol{n} = 0$$

in other words

$$(x - 1, y + 2, z - 3) \cdot (9, -3, 1) = 0$$

that is

$$9x - 3y + z = 9 + 6 + 3 = 18.$$

□

3.2 Directional Derivatives

The directional derivative of a scalar field in ϕ is its rate of change in the direction of a unit vector $\boldsymbol{u}$. To be more precise, consider a point with position vector $\boldsymbol{r}_0$. Then the line through P in the direction $\boldsymbol{u}$ has equation

$$\boldsymbol{r}(s) = \boldsymbol{r}_0 + s\boldsymbol{u}$$

Definition 3.7 The *directional derivative* of ϕ at the point P in the direction of the unit vector $\boldsymbol{u}$ is the derivative

$$\frac{d\phi(\boldsymbol{r}(s))}{ds}(0)$$

Proposition 3.8 *The above directional derivative is given by the expression*

$$\boldsymbol{\nabla}(\boldsymbol{r}_0)\phi \cdot \boldsymbol{u}$$

Proof: By the chain rule

$$\frac{d\phi(\boldsymbol{r}(s))}{ds} = \boldsymbol{\nabla}\phi(\boldsymbol{r}(s)) \cdot \boldsymbol{u}$$

Now $\boldsymbol{r}(0) = \boldsymbol{r}_0$, so if we set $s = 0$ we get

$$\frac{d\phi(\boldsymbol{r}(s))}{ds} = \boldsymbol{\nabla}\phi(\boldsymbol{r}_0) \cdot \boldsymbol{u}$$

as required. □

Example 3.9 Find the directional derivative of the vector field $\phi(x, y, z) = x^2y + xz$ in the direction $\boldsymbol{r} = (2, -2, 1)$ at the point $(1, 2, -1)$.

Solution: To find the directional derivative in a direction, we first need a unit vector pointing in that direction. A unit vector in the direction of $\boldsymbol{r}$ is gvien by

$$\boldsymbol{u} = \frac{1}{|\boldsymbol{r}|}\boldsymbol{r} = \frac{1}{\sqrt{2^2+2^2+1^1}}\boldsymbol{r} = \frac{1}{3}(2,-2,1)$$

We have

$$\boldsymbol{\nabla}\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = (2xy+z, x^2, x)$$

At $(1,2,-1)$ we have $\boldsymbol{\nabla}\phi(1,2,-1) = (3,1,1)$ so by the above we have directional derivative

$$\boldsymbol{\nabla}\phi(1,2,-1)\cdot\boldsymbol{u} = \frac{1}{3}(6-2+1) = \frac{5}{3}.$$

□

3.3 The Divergence and Curl of a Vector Field

Definition 3.10 Let $\boldsymbol{F}$ be a vector field. In components, write $\boldsymbol{F} = (F_1, F_2, F_3)$. Then we define the *divergence* of $\boldsymbol{F}$ to be the scalar field

$$div\,\boldsymbol{F} = \boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

We define the *curl* of $\boldsymbol{F}$ to be the vector field

$$curl\,\boldsymbol{F} = \boldsymbol{\nabla}\times\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Both div and curl are important in formulating Maxwell's equations of electrodynamics.

Example 3.11 Let $\boldsymbol{F}(x,y,z) = (x^2, xy, yz)$. Find the divergence and curl of $\boldsymbol{F}$.

Solution: We have $F_1 = x^2$, $F^2 = xy$ and $F_3 = yz$. So

$$\frac{\partial F_1}{\partial x} = 2x \qquad \frac{\partial F_2}{\partial y} = x \qquad \frac{\partial F_3}{\partial z} = y$$

and

$$div\,\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2x + x + y = 3x + y.$$

Now

$$curl\,\boldsymbol{F} = \boldsymbol{\nabla}\times\boldsymbol{F} == \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

that is to say

$$\begin{aligned} curl\,\boldsymbol{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k} \\ &= (z-0)\boldsymbol{i} + (0-0)\boldsymbol{j} + (y-0)\boldsymbol{k} = (z, 0, y). \end{aligned}$$

□

Example 3.12 Let $\boldsymbol{r}(x, y, z) = (x, y, z)$ as usual. Then $div\,\boldsymbol{r} = 3$ and $curl\,\boldsymbol{r} = \boldsymbol{0}$.

Example 3.13 Consider water flowing in circular paths, such as water draining from a sink. A small volume of water at the point (x, y, z) at time t has position vector $\boldsymbol{r}(t) = (r\cos(\omega t), r\sin(\omega t), 0)$. Calculate the velocity vector field $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$ and determine $div\,\boldsymbol{v}$ and $curl\,\boldsymbol{v}$.

Solution: We have

$$\boldsymbol{v} = (-r\omega\sin(\omega t), r\omega\cos(\omega t), 0) = \omega(-y, x, 0).$$

Now

$$div\,\boldsymbol{v} = \boldsymbol{\nabla}\cdot\boldsymbol{F} = 0 + 0 + 0 = 0$$

and

$$curl\,\boldsymbol{v} = \boldsymbol{\nabla}\times\boldsymbol{F} == \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix}$$

that is to say

$$curl\,\boldsymbol{v} = (0 = 0)\boldsymbol{i} - (0-0)\boldsymbol{j} + (\omega - (-\omega))\boldsymbol{k} = 2\omega\boldsymbol{k}.$$

□

Recall that we call a scalar or vector field *smooth* of partial derivatives of all orders exist, and this property means we can swap the order of taking partial derivatives.

Proposition 3.14 *Let ϕ be a smooth differentiable scalar field. Then* $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) = \boldsymbol{0}$.

Proof: We have $\boldsymbol{\nabla}\phi = (\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z})$. Observe

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

that is

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi) = \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}, \cdots, \cdots \right) = (0, 0, 0).$$

□

In other words, $curl(grad\, \phi) = \mathbf{0}$. We call a vector field $\boldsymbol{F}$ such that $curl\, \boldsymbol{F} = \mathbf{0}$ *irrotational*. If $\boldsymbol{F}$ is a force, we call it *conservative*. The remarkable thing is that a converse to the above proposition is true.

Theorem 3.15 *Let $\boldsymbol{F}$ be an irrotational vector field. Then there is a scalar field ϕ such that $\boldsymbol{\nabla}\phi = \boldsymbol{F}$.*

We shall prove this result later on when we have introduced line integrals. We call ϕ a *scalar potential* for $\boldsymbol{F}$.

The following proposition is proved similarly to the above.

Proposition 3.16 *Let ϕ be a smooth differentiable vector field. Then $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) = 0$.* □

In other words, $div(curl\, \boldsymbol{F}) = 0$. Again we have a harder to prove converse.

Theorem 3.17 *Let $\boldsymbol{\nabla} \cdot \boldsymbol{G} = 0$. Then there is a scalar field $\boldsymbol{F}$ such that we can write $\boldsymbol{G} = \boldsymbol{\nabla} \times \boldsymbol{F}$.* □

We call the vector field $\boldsymbol{F}$ the *vector potential* for $\boldsymbol{G}$. For example, in electromagnetism, the magnetic induction field $\boldsymbol{B}$ satisfies the equation $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$. Therefore there exists a vector potential $\boldsymbol{A}$ such that $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$.

3.4 The Laplacian

Definition 3.18 The *Laplacian* of a scalar field $\phi(x, y, z)$ is the scalar field $\nabla^2\phi$ defined by writing

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}.$$

]

The symbol ∇^2 stands for

$$\nabla^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

We can also define the Laplacian of a vector field $\boldsymbol{F}(x, y, z)$

$$\nabla^2 \boldsymbol{F} = \frac{\partial^2 \boldsymbol{F}}{\partial x^2} + \frac{\partial^2 \boldsymbol{F}}{\partial y^2} + \frac{\partial^2 \boldsymbol{F}}{\partial z^2}.$$

The equation $\nabla^2\phi = 0$ for a scalar field ϕ is called *Laplace's equation*. It has several important applications in different phenomena in mathematical physics, such as fluids and elecromagnetism.

Example 3.19 The gravitational potential of a spherical mass M, for points outside the mass is defined by

$$\phi(\boldsymbol{r}) = \frac{GM}{|\boldsymbol{r}|}$$

where G is Newton's gravitational constant, and the origin is at the centre of the mass. Find $\boldsymbol{\nabla}\phi$ in its simplest form, and show that ϕ satisfies Laplace's equation.

Solution: Let $r = |\boldsymbol{r}| = \sqrt{x^2+y^2+z^2}$. Then

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

so by the ordinary chain rule for differentiation

$$\frac{\partial \phi}{\partial x} = GM\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -GM\frac{1}{r^2}\cdot\frac{x}{r} = -\frac{GMx}{r^3}.$$

Similarly

$$\frac{\partial \phi}{\partial y} = -\frac{GMy}{r^3} \qquad \frac{\partial \phi}{\partial z} = -\frac{GMz}{r^3}$$

Thus

$$\boldsymbol{\nabla}\phi = -\frac{GMx}{r^3}\boldsymbol{i} - \frac{GMx}{r^3}\boldsymbol{j} - \frac{GMx}{r^3}\boldsymbol{k} = -\frac{GM\boldsymbol{r}}{r^3} = -\frac{GM}{r^2}\hat{\boldsymbol{r}}.$$

where $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$ is the unit vector in the direction of $\boldsymbol{r}$.

Now by the quotient rule

$$\frac{\partial^2 \phi}{\partial x^2} = -GM\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = -GM\frac{r^3 - 3r^2x\frac{x}{r}}{r^6} = -GM\frac{r^3-3rx^2}{r^6} = -GM\frac{r^2-3x^2}{r^5}$$

Similarly

$$\frac{\partial^2 \phi}{\partial y^2} = -GM\frac{r^2-3y^2}{r^5} \qquad \frac{\partial^2 \phi}{\partial z^2} = -GM\frac{r^2-3z^2}{r^5}$$

Adding:

$$\nabla^2\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = -GM\frac{r^2+r^2+r^2-3(x^2+y^2+z^2)}{r^5} = 0$$

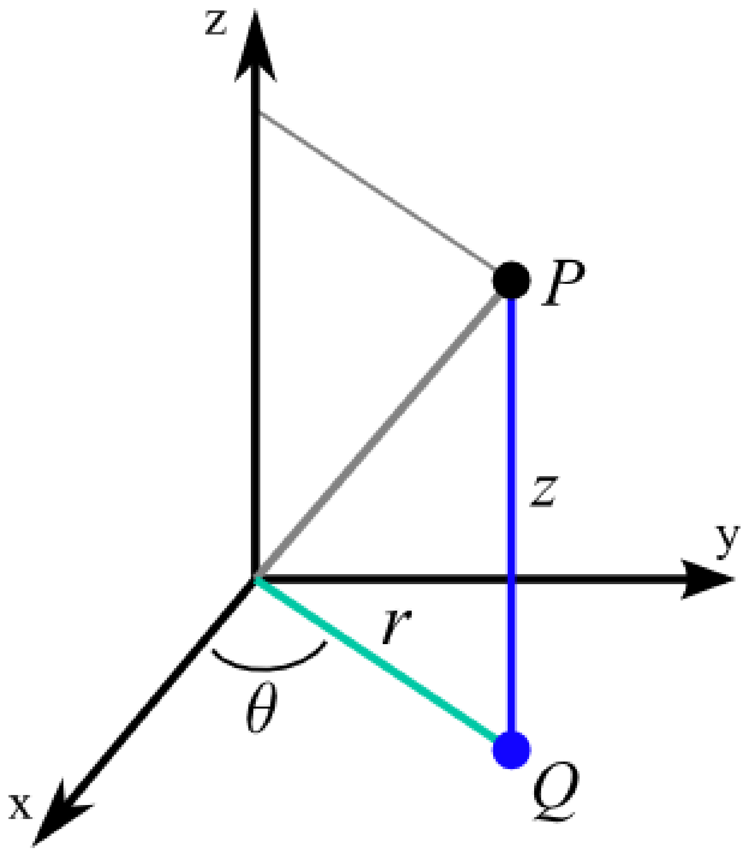
since $x^2+y^2+z^2 = r^2$.

Hence Laplace's equation holds for ϕ. □

3.5 Cylindrical Polar Coordinates

In *cylindrical polar coordinates*, a point P with position vector (x, y, z) is represented by three coordinates r, θ and z where $r \geq 0$, $0 \leq \theta \leq 2\pi$, and

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z \text{ as before}$$



We see that

$$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

so $r = \sqrt{x^2 + y^2}$ (*not* $r = \sqrt{x^2 + y^2 + z^2}$). These coordinates are useful for three-dimensional problems where there is rotational symmetry about an axis.

Let $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{z}}$ be unit vectors in the direction of increasing $\boldsymbol{r}$, $\boldsymbol{\theta}$ and $\boldsymbol{z}$ respectively. We can express grad, div and curl in terms of cylindrical polar coordinates, and these vectors. This is just a calculation; we omit the proofs.

Proposition 3.20 *Let V be a scalar field. Then*

$$\boldsymbol{\nabla} V = \frac{\partial V}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial V}{\partial z}\hat{\boldsymbol{z}}$$

□

Example 3.21 Let V be a scalar field expressed in cylindrical polar coordinates (r, θ, z) by the formula

$$V = \frac{\sin 2\theta}{r^2} + \frac{2(z-1)}{r^3}.$$

Calculate $\boldsymbol{\nabla} V$.

Solution: We have

$$\boldsymbol{\nabla} V = \frac{\partial V}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial V}{\partial z}\hat{\boldsymbol{z}}$$

Now

$$\frac{\partial V}{\partial r} = -2\frac{\sin 2\theta}{r^3} - \frac{6(z-1)}{r^4}$$

and

$$\frac{\partial V}{\partial \theta} = 2\frac{\cos 2\theta}{r^2} \qquad \frac{\partial V}{\partial z} = \frac{2}{r^3}$$

so

$$\boldsymbol{\nabla} V = -\left(\frac{2\sin 2\theta}{r^3} + \frac{6(z-1)}{r^4}\right)\hat{\boldsymbol{r}} + \frac{2\cos 2\theta}{r^3}\hat{\boldsymbol{\theta}} + \frac{2}{r^3}\hat{\boldsymbol{z}}.$$

□

Proposition 3.22 *Let $\boldsymbol{F}$ be a vector field, expressed in cylindrical polar coordinates as*

$$\boldsymbol{F} = F_1\hat{\boldsymbol{r}} + F_2\hat{\boldsymbol{\theta}} + F_3\hat{\boldsymbol{z}}.$$

Then

$$\boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_1) + \frac{1}{r}\frac{\partial F_2}{\partial \theta} + \frac{\partial F_3}{\partial z}$$

and

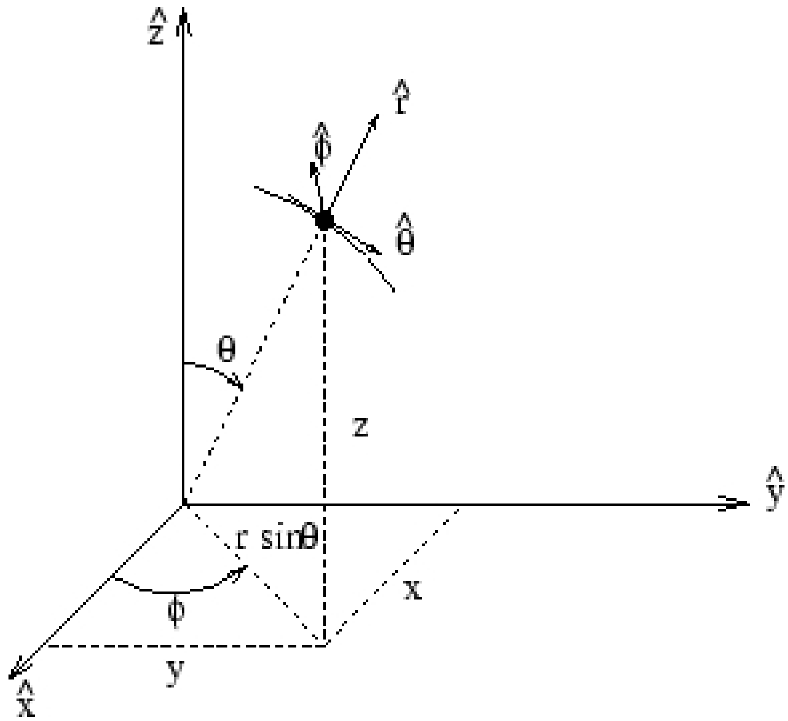
$$\boldsymbol{\nabla}\times\boldsymbol{F} = \frac{1}{r}\begin{vmatrix} \hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_1 & rF_2 & F_3 \end{vmatrix}$$

□

3.6 Spherical Polar Coordinates

In *spherical polar coordinates*, a point P with position vector (x, y, z) is represented by three coordinates r, θ and ϕ where $r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$



We see that

$$\begin{aligned}
x^2+y^2+z^2 &= r^2\sin^2\theta(\cos^2\phi+\sin^2\phi)+r^2\cos^2\theta \\
&= r^2\sin^2\theta+r^2\cos^2\theta \\
&= r^2
\end{aligned}$$

so $r=\sqrt{x^2+y^2+z^2}$. These coordinates are useful for three-dimensional problems where there is spherical symmetry.

Let $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ be unit vectors in the direction of increasing $\boldsymbol{r}$, $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$ respectively. We can express grad, div and curl in terms of spherical polar coordinates, and these vectors. This is just a calculation; we omit the proofs.

Proposition 3.23 *Let V be a scalar field. Then*

$$\boldsymbol{\nabla} V=\frac{\partial V}{\partial r}\hat{\boldsymbol{r}}+\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}}+\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\boldsymbol{\phi}}$$

□

Example 3.24 Let V be a scalar field expressed in sphericall polar coordinates (r,θ,z) by the formula

$$V=\frac{\sin\theta\cos\phi}{r}+\frac{2\cos\theta\cos\phi}{r^2}.$$

Calculate $\boldsymbol{\nabla} V$.

Solution: We have

$$\boldsymbol{\nabla} V = \frac{\partial V}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Now

$$\frac{\partial V}{\partial r} = -\frac{\sin\theta\cos\phi}{r^2} - \frac{4\cos\theta\cos\phi}{r^3}$$

$$\frac{\partial V}{\partial \theta} = \frac{\cos\theta\cos\phi}{r} - \frac{2\sin\theta\cos\phi}{r^2}$$

and

$$\frac{\partial V}{\partial \phi} = -\frac{\sin\theta\sin\phi}{r} - \frac{2\cos\theta\sin\phi}{r^2}$$

so

$$\boldsymbol{\nabla} V = \left(\frac{\sin\theta\cos\phi}{r^2} + \frac{4\cos\theta\cos\phi}{r^3}\right)\hat{\boldsymbol{r}} + \left(\frac{|cos\theta\cos\phi}{r} - \frac{2\sin\theta\cos\phi}{r^2}\right)\hat{\boldsymbol{\theta}} - \left(\frac{\sin\theta\sin\phi}{r} + \frac{2\cos\theta\sin\phi}{r^2}\right)\hat{\boldsymbol{\phi}}$$

□

Proposition 3.25 *Let $\boldsymbol{F}$ be a vector field, expressed in spherical polar coordinates as*

$$\boldsymbol{F} = F_1\hat{\boldsymbol{r}} + F_2\hat{\boldsymbol{\theta}} + F_3\hat{\boldsymbol{\phi}}.$$

Then

$$\boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(F_2\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial F_3}{\partial\phi}$$

and

$$\boldsymbol{\nabla}\times\boldsymbol{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ F_1 & rF_2 & r\sin\theta F_3 \end{vmatrix}$$

□

4 Double Integrals

4.1 Repeated Integrals

Consider a function of two variables, $f(x, y)$ defined for $a \le x \le b$ and $c \le y \le d$. Then we can form the *repeated integral*

$$I = \int_c^d \int_a^b f(x, y)\ dx\ dy$$

This means integrate first with respect to x, treating y as a constant, to get a function of y, then with respect to y. In other words

$$I = \int_c^d \left(\int_a^b f(x,y)\ dx \right)\ dy$$

This makes sense as the integral $\int_a^b f(x,y)\ dx$ is a function of y, $g(y)$, and

$$I = \int_c^d g(y)\ dy$$

In some books, this double integral is written as

$$I = \int_c^d dy \int_a^b f(x,y)\ dx$$

though we do not use this notation in this course.

Example 4.1 Evaluate

$$I = \int_1^2 \int_0^1 (2x+6y)\ dx\ dy$$

Solution: We have

$$I = \int_1^2 \int_0^1 (2x+6y)\ dx\ dy = \int_1^2 \left[x^2+6xy\right]_0^1\ dy$$

that is

$$I = \int_1^2 1+6y\ dy = \left[y+2y^2\right]_1^2 = 2+12-(1+3) = 10.$$

□

It turns out that if we also evaluate

$$J = \int_0^1 \int_1^2 (2x+6y)\ dy\ dx$$

(ie: integrate first with respect to y, and then with respect to x), we also get the answer 10. This is not a coincidence.

Theorem 4.2 *Let $f(x,y)$ be a continuous function, where $a \le x \le b$ and $c \le y \le d$. Then*

$$\int_c^d \int_a^b f(x,y)\ dx\ dy = \int_a^b \int_c^d f(x,y)\ dy\ dx$$

□

Sometimes we can 'separate variables' in repeated integrals.

Proposition 4.3

$$\int_c^d \int_a^b f(x)g(y)\ dx\ dy = \left(\int_a^b f(x)\ dx\right)\left(\int_c^d g(y)\ dy\right).$$

Proof: Let
$$I = \int_c^d \int_a^b f(x)g(y)\ dx\ dy.$$

Note that with respect to x, $g(y)$ is a constant, so

$$\int_a^b f(x)g(y)\ dx = \left(\int_a^b f(x)\ dx\right) g(y).$$

Now, $\int_a^b f(x)\ dx$ is just a fixed number, meaning it is a constant as y varies, so
$$I = \int_c^d \left(\int_a^b f(x)\ dx\right) g(y)\ dy = \left(\int_a^b f(x)\ dx\right)\left(\int_c^d g(y)\ dy\right).$$

□

Example 4.4 Evaluate

$$I = \int_0^1 \int_0^1 e^{x+y}\ dx\ dy.$$

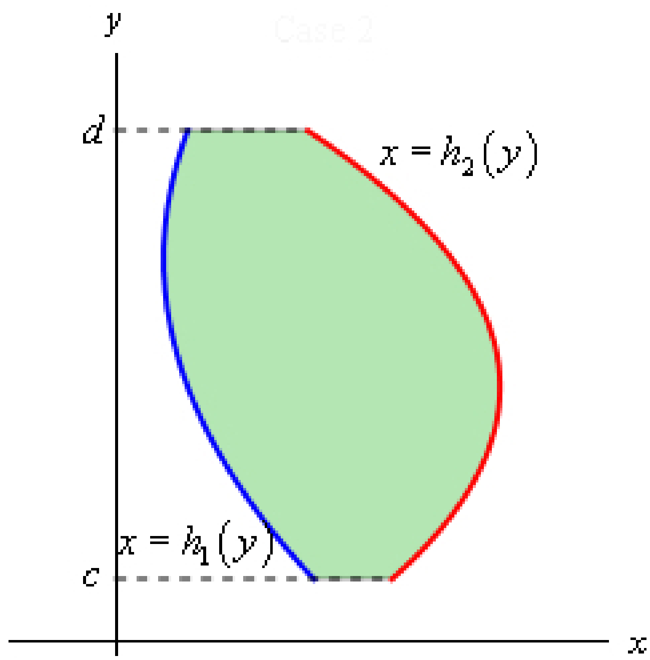
Solution: We have $e^{x+y} = e^x e^y$, so by the above

$$I = \left(\int_0^1 e^x\ dx\right)\left(\int_0^1 e^y\ dy\right) = \left(\int_0^1 e^x\ dx\right)^2 = \left([e^x]_0^1\right)^2 = (e-1)^2.$$

□

4.2 Area Integrals

So far we have only integrated a function $f(x, y)$ over a recrangular area. But we can generalise to integrating a function $f(x, y)$ where (x, y) belongs to a region, R, as shown.

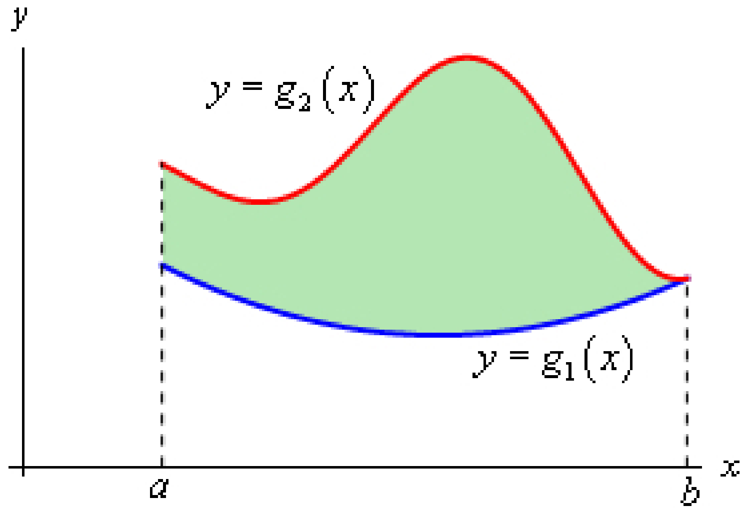


Thus, in this case we have $h_1(y) \leq x \leq h_2(y)$ and $c \leq y \leq d$. We write this integral

$$\int\int_R f(x,y)\ dx\ dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\ dx\ dy$$

Again, we evaluate first by integrating with respect to x and then with respect to y. The limits of the first integral are not constants, but functions of y.

For certain shaped regions we are better off integrating first with respect to y and then with respect to x:



We see that here we have $g_1(x) \leq y \leq g_2(y)$ and $a \leq x \leq b$. Our integral is

$$\int \int_R f(x,y)\ dx\ dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dy\ dx$$

Some remarks before we look at examples:

- We need to be careful with the limits of the integral when we swap the order of integration. In particular, it is *not* true that

  $$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\ dx\ dy = \int_{h_1(y)}^{h_2(y)} \int_c^d f(x,y)\ dy\ dx$$

  Indeed, this equation is nonsense. The left hand side is just a number (as we would expect). The right hand side is a function of y.

- If the region R is not convex, it makes more sense to do the integration in a certain order. Thus in the second of our examples, the first (inner) integral is with respect to y. The shape of the region means this makes less sense to do first with respect to x; we would have to divide the region into pieces and add them.

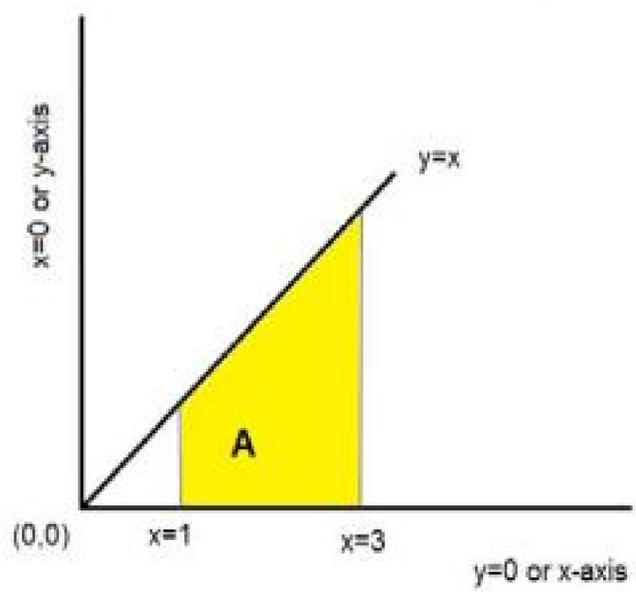
- It sometimes *really* helps to draw a picture. I suggest doing this every time.

Example 4.5 Evaluate the integral

$$I = \int\int_R 3x^2 + 3y^2 \; dx \; dy$$

where R is the region where $1 \le x \le 3$ and $0 \le y \le x$.

Solution: First, as suggested, we draw a picture of the domain.



We see we need to integrate first with respect to y. So

$$I = \int\int_R = \int_1^3 \int_0^x 3x^2 + 3y^2 \; dy \; dx = \int_1^3 \left[3x^2 y + y^3\right]_0^x \; dx = \int_1^3 3x^3 + x^3 \; dx$$

that is

$$I = \int_1^3 4x^3 \; dx = \left[x^4\right]_1^3 = 81 - 1 = 80.$$

□

Example 4.6 Evaluate the integral

$$I = \int\int_R 4xy \; dx \; dy$$

where R is the region in between the two graphs

$$y = 2 + x \qquad y = x^2 - 6x + 6.$$

Solution: First let us work out where the graphs intersect. This happens at values of x where

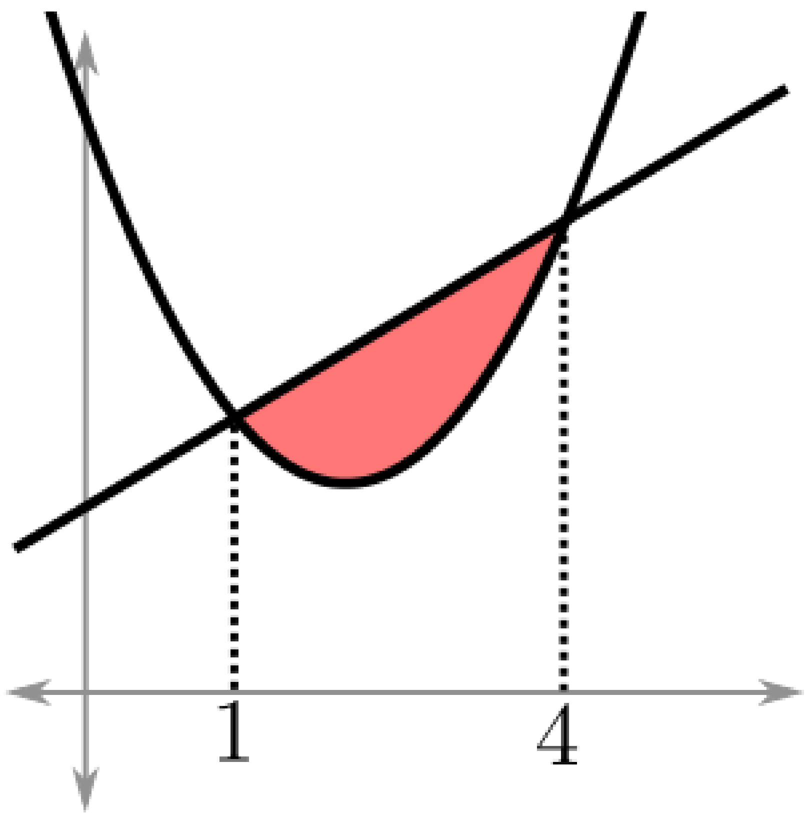
$$2 + x = x^2 - 6x + 6$$

that is

$$0 = x^2 - 5x + 4 = (x - 1)(x - 4)$$

so when $x = 1$ and $x = 4$.

Here we really need a picture:



Thus R is the region where $1 \leq x \leq 4$ and $2 + x \leq y \leq x^2 - 6x + 6$. Again we need to integrate first with respect to y, so

$$I = \int_1^4 \int_{x^2-6x+6}^{2+x} 4xy \; dy \; dx = \int_1^4 \left[2xy^2\right]_{x^2-6x+6}^{2+x} \; dx$$

Now

$$\begin{aligned} \left[2xy^2\right]_{x^2-6x+6}^{2+x} &= 2x(2+x)^2 - 2x(x^2 - 6x + 6)^2 \\ &= 2x(4 + 4x + x^2 - x^4 - 36x^2 - 36 - 12x^2 + 12x^3 + 72x) \\ &= -64x + 152x^2 - 94x^3 + 24x^4 - 2x^5 \end{aligned}$$

Hence

$$I = \int_1^4 -64x + 152x^2 - 94x^3 + 24x^4 - 2x^5 \; dx = \left[-32x^2 + \frac{152}{3}x^3 + \frac{47}{2}x^4 + \frac{24}{5}x^5 - \frac{1}{3}x^6 \right]_1^4 = 7539.5.$$

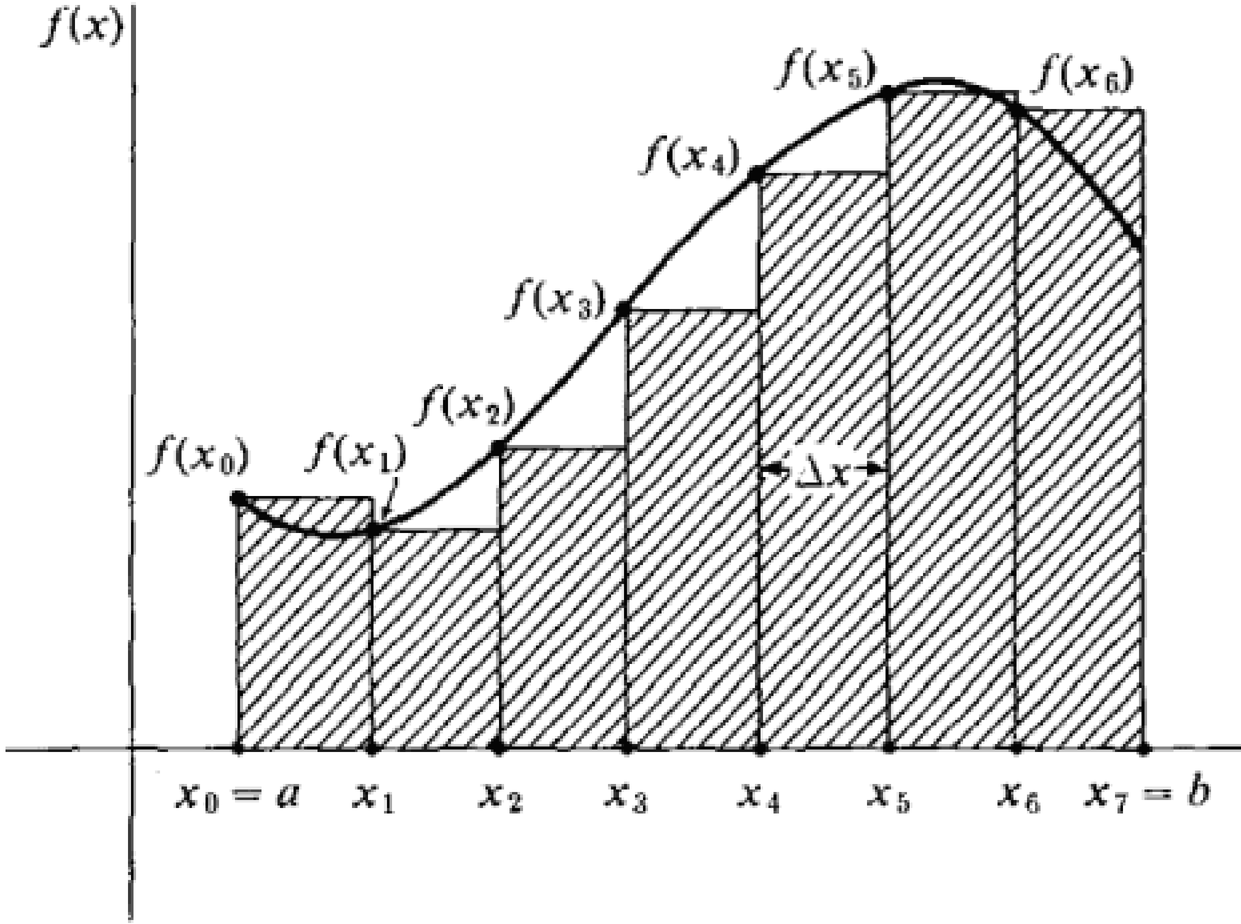
□

4.3 The Meaning of the Integral

Let $f(x) > 0$ for $a \leq x \leq b$. Then

$$I = \int_a^b f(x) \; dx$$

is the area under the graph of f.



This can be approximated by dividing the area into small rectangles, each of width Δx, as shown. The height of a rectangle poisitioned at x is $f(x)$.

Thus the area of the rectangle positioned at x is $f(x)\ \Delta x$, and if there are n rectangles, we see that the area is approximately

$$\sum_{i=1}^{n} f(x_i)\ \Delta x.$$

The approximation becomes exact as $\Delta x \to 0$. In other words,

$$\sum_{i=1}^{n} f(x_i)\ \Delta x \to \int_a^b f(x)\ dx$$

as $\Delta x \to 0$. Mathematicians use this as the *definition* of the integral.

A similar logic applies to double integrals. For a function $f(x, y) > 0$, for $a \leq x \leq b$ and $c \leq y \leq d$, the double integral

$$\int_c^d \int_a^b f(x, y)\ dx\ dy$$

is the *volume* under the function $f(x, y)$.

If we divide the volume into cuboids, with base dimensions Δx and Δy, then the height of the cuboid positioned at (x, y) is $f(x, y)$, and its volume is $f(x, y)\ \Delta x\ \Delta y$. If there are m cuboids in the x-direction, and n in the y-direction, then the volume is approximately

$$\sum_{i,j=1}^{m,n} f(x_i, y_j) \Delta x\ \Delta y.$$

The approximation becomes exact as $\Delta x \to 0$. In other words,

$$\sum_{i=1}^{n} f(x_i, y_j)\ \Delta x\ \ \to \int_c^d \int_a^b f(x, y)\ dx\ dy$$

as $\Delta x,\ \Delta y \to 0$.

Note that this explains why we can swap around the order of a double integral.

The above can be generalised to the integral over a region R, ie:

$$\sum_{(x_i, y_j) \in R} f(x_i, y_j)\ \Delta x \Delta y \to \int \int_R f(x, y)\ dx\ dy$$

as $\Delta x,\ \Delta y \to 0$.

Having an idea of what the integral means is useful in proving theorems, working out formulae, and using integrals of various kinds in physical applications. Here, what is important is the shape of a region R, and the so-called *element of area*

$$dA = dx\ dy$$

which is approximated by

$$\Delta A = \Delta x\ \Delta y$$

where the approximation becomes exact as $\Delta x,\ \Delta y \to 0$.

Example 4.7 Suppose we have a plane lamina, occupying the region R of the (x, y)-plane, and suppose the surface density (ie: mass per unit area) is given by the function $\sigma(x, y)$. What is the mass of the lamina?

Solution: The mass of an element of area, ΔA, at (x, y), is approximately $\Delta A = \sigma(x, y)\ \Delta x\ \Delta y$. Hence, the total mass M is approximately

$$M \approx \sum_{(x_i,y_j)\in R} \sigma(x_i, y_j)\ \Delta x\ \Delta y.$$

Let $\Delta x,\ \Delta y \to 0$. Then the approximation becomes exact, and

$$M = \int\int_R \sigma(x_i, y_j)\ dx\ dy.$$

□

4.4 Plane Polar Coordinates

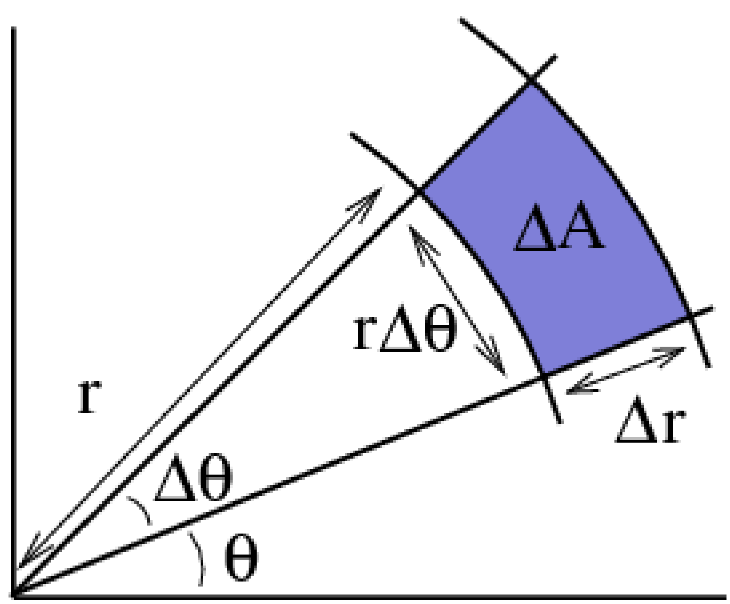
Sometimes, for functions or regions with certian symmetries, we want to work out an area integral

$$I = \int\int_R f(x, y)\ dx\ dy$$

in terms of polar coordinates r and θ, where as usual $x = r\cos\theta$ and $y = r\sin\theta$. Let

$$g(r, \theta) = f(r\cos\theta, r\sin\theta).$$

In plane polar coordinates a small area ΔA is given by $\Delta A \approx r\ \Delta r\ \Delta\theta$.



This tells us that

$$I \approx \sum_{(r_i,\theta_j)\in R} f(r_i \cos\theta_j, r_i \sin\theta_j) r \ \delta r \ \delta\theta$$

so if we let $\Delta r, \ \Delta\theta \to 0$ we see

$$I = \int \int_R g(r,\theta) r \ dr \ d\theta.$$

The most elegant way to express this is to introduce the *element of area*. Here, an area integral is an integral of the form

$$I = \int \int_R f(x,y) \ dA$$

where I is a two-dimensional region, and we write $dA = dx \ dy$. The above formula can then by expressed simply by saying

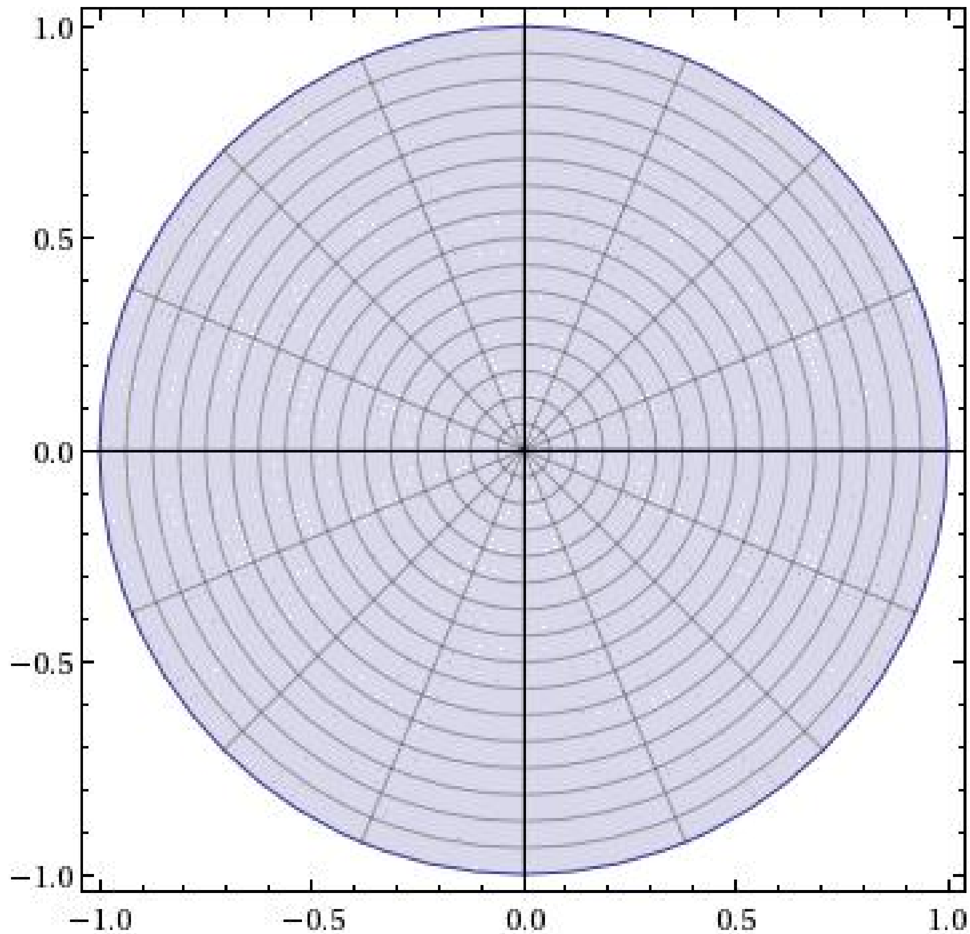
$$dA = r \ dr \ d\theta.$$

Example 4.8 Calculate the integral

$$\int \int_R x^2 + y^2 \ dx \ dy$$

where R is region where $x^2 + y^2 \leq 1$.

Solution: The region R is the unit disk



which is given in plane polar coordinates (r, θ) by $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Further, $x^2 + y^2 = r^2$. Hence

$$\int\int_R x^2{+}y^2\ dx\ dy = \int_0^{2\pi}\int_0^1 (r^2) r\ dr\ d\theta = \int_0^{2\pi}\int_0^1 r^3\ dr\ d\theta = \int_0^{2\pi}\left[\frac{1}{4}r^4\right]_0^1 d\theta = \frac{\pi}{2}.$$

□

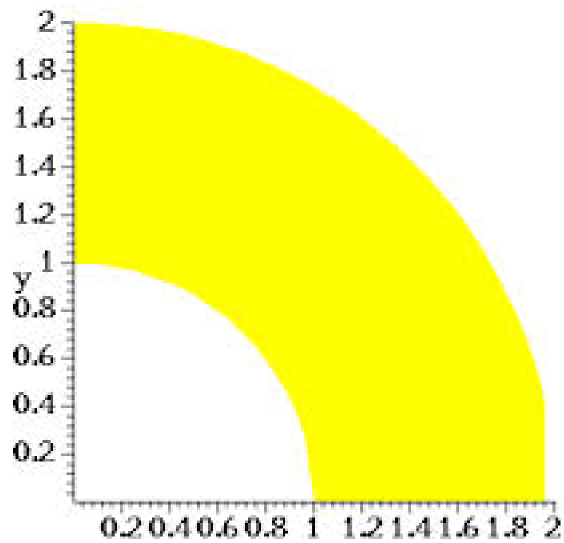
Example 4.9 Calculate the integral

$$I = \int\int_R x\ dx\ dy$$

where R is the region where $x \geq 0$, $y \geq 0$ and

$$1 \leq x^2 + y^2 \leq 4$$

Solution: The region R is the quarter circle shown:



In plane polar coordinates (r, θ) this region is described by the inequalities $1 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{2}$. Note that $x = r \cos\theta$. So

$$I = \int_0^{\frac{\pi}{2}} \int_1^2 (r \cos\theta) r \; dr \; d\theta = \int_0^{\frac{\pi}{2}} \int_1^2 r^2 \cos\theta \; dr \; d\theta = \int_0^{\frac{\pi}{2}} \left[\frac{1}{3} r^3 \right]_1^2 d\theta = \frac{7\pi}{6}$$

□

5 Integrals and Vector Calculus

5.1 Line Integrals

A *smooth curve*, C, is a vector valued function $\boldsymbol{r}(t)$, defined for $a \leq t \leq b$ such that the derivative $\boldsymbol{r}'(t) = \frac{d\boldsymbol{r}}{dt}$ exists for all $a \leq t \leq b$, and is continuous. We call $\boldsymbol{r}(t)$ the function that *parametrises* C.

The derivative $\boldsymbol{r}'(t)$ is the tangent vector to the curve.

Definition 5.1 Let $\boldsymbol{F}$ be a vector field, and let C be a smooth curve, parametrised by a function $\boldsymbol{r}(t)$. Then we define the *line integral* of $\boldsymbol{F}$ along C by the equation

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \frac{d\boldsymbol{r}}{dt} \; dt$$

Note that this integral is a *scalar*. In terms of components, let

$$\boldsymbol{F} = (F_1, F_2, F_3) \qquad \boldsymbol{r}(t) = (x(t), y(t), z(t))$$

Then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b F_1 \frac{dx}{dt}\ dt + \int_a^b F_2 \frac{dy}{dt}\ dt + \int_a^b F_3 \frac{dz}{dt}\ dt$$

We also use the notation

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C F_1\ dx + F_2\ dy + F_3\ dz$$

to denote the path integral.

The following is fairly obvious.

Proposition 5.2 *If C is a curve made up of joining two curves C_1 and C_2 together, then*

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

□

We can use the above to define path integrals when C is not a smooth curve, but is made up of joining smooth curves together.

Example 5.3 Evaluate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where C is the path consisting of a straight line from $(1, -1, 0)$ to $(1, 1, 0)$, followed by a straight line from $(1, 1, 0)$ to $(-2, 1, 0)$ and $\boldsymbol{F}(x, y, z) = (xy^2, x - y, 0)$.

Solution: Note that C is a join of two curves, C_1 and C_2, where C_1 is the straight line from $(1, -1, 0)$ to $(1, 1, 0)$, and C_2 is the straight line from $(1, 1, 0)$ to $(-2, 1, 0)$.

We can parametrise C_1 by:

$$\boldsymbol{r}(t) = (1, t, 0) \qquad -1 \leq t \leq 1$$

ie:

$$x = 1 \qquad y = t \qquad -1 \leq t \leq 1.$$

Observe

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{j} = (0, 1, 0).$$

Since $\boldsymbol{F} = (xy^2, x - y, 0)$, we have

$$\boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{dt} = x - y = 1 - t$$

and

$$\int_{C_1} \boldsymbol{F} \cdot \boldsymbol{r} = \int_{-1}^{1} \boldsymbol{F} \cdots \frac{\boldsymbol{r}}{dt}\ dt = \int_{-1}^{1} 1 - t\ dt = \left[t - \frac{1}{2}t^2\right]_{-1}^{1} = 2.$$

We can parametrise C_2 by:

$$\boldsymbol{r}(t) = (-t, 1, 0) \qquad -1 \leq t \leq 2$$

ie:

$$x = -t \qquad y = 1 \qquad -1 \leq t \leq 2.$$

Observe

$$\frac{d\boldsymbol{r}}{dt} = -\boldsymbol{i} = (1, 0, 0).$$

Since $\boldsymbol{F} = (xy^2, x - y, 0)$, we have

$$\boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{dt} = xy^2 = -t$$

and

$$\int_{C_2} \boldsymbol{F} \cdot \boldsymbol{r} = \int_{-1}^{2} \boldsymbol{F} \cdots \frac{\boldsymbol{r}}{dt}\ dt = \int_{-1}^{2} t\ dt = \left[\frac{1}{2}t^2\right]_{-1}^{2} = \frac{3}{2}.$$

Hence

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} = \frac{7}{2}.$$

□

From physics, we know that the *work* done by a force $\boldsymbol{F}$ which undergoes a small displacement $\delta\boldsymbol{r}$ is

$$\delta W = \boldsymbol{F} \cdot \delta\boldsymbol{r}$$

Hence, if a particle moves along a curve C, parametrised by $\boldsymbol{r}(t)$, $a \leq t \leq b$, in a force given by a vector field $\boldsymbol{F}$, then the total work done is, approximately given by a sum of the form

$$\sum_{i=1}^{n} \boldsymbol{F}(\boldsymbol{r}(t_i)) \cdot \delta\boldsymbol{r}(t_i) = \sum_{i=1}^{n} \boldsymbol{F}(\boldsymbol{r}(t_i)) \cdot \frac{\delta\boldsymbol{r}(t_i)}{\delta t}\delta t$$

Let $\delta t \to 0$. Then we see that the total work is

$$W = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \frac{d\boldsymbol{r}}{dt}\ dt = \int_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

Example 5.4 Evaluate

$$\int_C \boldsymbol{F} \cdot \boldsymbol{r}$$

where $\boldsymbol{F} = (y^2, 2xy + z, y)$ for the following curves C joining the point $(0, 0, 0)$ and $(1, 1, 2)$.

- C is given by the equation $\boldsymbol{r} = (2t^2 - t, t, 2t^2)$, where $0 \leq t \leq 1$.
- C is the straight line from $(0, 0, 0)$ to $(1, 1, 2)$.

Solution:

- Observe that
$$\frac{d\boldsymbol{r}}{dt} = (4t-1, 1, 4t)$$
On C, we have $x = 2t^2 - t$, $y = t$ and $z = 2t^2$, so
$$\boldsymbol{F}(\boldsymbol{r}(t)) = (y^2, 2xy + z, y) = (t^2, 2(2t^2 - t)t + 2t^2, t) = (t^2, 4t^3, t)$$
and
$$\int_C \boldsymbol{F} \cdot \boldsymbol{r} = \int_0^1 (t^2, 4t^3, t) \cdot (4t-1, 1, 4t)\ dt = \int_0^1 4t^3 - t^2 + 4t^3 + 4t^2\ dt$$
that is
$$\int_C \boldsymbol{F} \cdot \boldsymbol{r} = \int_0^1 8t^3 + 3t^2\ dt = \left[2t^4 + t^3\right]_0^1 = 2 + 1 = 3$$

- In this case, C is given by the equation
$$\boldsymbol{r}(t) = (t, t, 2t) \qquad 0 \leq t \leq 1.$$
So
$$\frac{\boldsymbol{r}}{dt} = (1, 1, 2)$$
On C, we have $x = t$, $y = t$ and $z = 2t$, so
$$\boldsymbol{F}(\boldsymbol{r}(t)) = (y^2, 2xy + z, y) = (t^2, 2t^2 + 2t, t)$$
and
$$\int_C \boldsymbol{F} \cdot \boldsymbol{r} = \int_0^1 (1, 1, 2) \cdot (t^2, 2t^2 + 2t, t)\ dt = \int_0^1 t^2 + 2t^2 + 2t + 2t\ dt$$
that is
$$\int_C \boldsymbol{F} \cdot \boldsymbol{r} = \int_0^1 3t^2 + 4t\ dt = \left[t^3 + 2t^2\right]_0^1 = 2 + 1 = 3$$

- In this case, C is given by the equation
$$\boldsymbol{r}(t) = (t, t, 2t) \qquad 0 \leq t \leq 1.$$

$\square$

In the above example, the path integral along a curve C does not appear to depend on C, but only on its endpoints. The exact conditions for independence of choice of curve are in the following theorem.

Theorem 5.5 *Let* $\boldsymbol{F}$ *be a vector field. Then the line integral* $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ *is independent of the curve* C*, depending only on its end points, if and only if* $\boldsymbol{F} = \boldsymbol{\nabla}\phi$ *for some scalar potential* ϕ*.*

Proof: If $\boldsymbol{F} = \boldsymbol{\nabla}\phi$, then we have

$$\boldsymbol{F} = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right).$$

Let C be parametrised by $\boldsymbol{r}(t)$, with $a \leq t \leq b$, and write

$$\boldsymbol{r}(t) = (x(t), y(t), z(t)).$$

Then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial t}\frac{dz}{dt}\ dt$$

so by the chain rule

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b \frac{d}{dt}(\phi(\boldsymbol{r}(t)))\ dt = \phi(\boldsymbol{r}(b)) - \phi(\boldsymbol{r}(a))$$

which depends only on the end points, $\boldsymbol{r}(b)$ and $\boldsymbol{r}(a)$, and not on the curve C.

Conversely, suppose the integral

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

depends only on $\boldsymbol{F}$ and the endpoints of the curve C.

For a point with position vector $\boldsymbol{r}$, let $C(\boldsymbol{r})$ be *any* curve from the origin 0 to $\boldsymbol{r}$. Define

$$\phi(\boldsymbol{r}) = \int_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

Then we can check that $\boldsymbol{\nabla}\phi = \boldsymbol{F}$, and we are done. □

Let $\phi(x, y, z) = xy^2 + yz$. Then

$$\boldsymbol{\nabla}\phi = (y^2, 2xy + z, y)$$

which is $\boldsymbol{F}$ in the above example.

5.2 Closed Curves and Green's Theorem

Definition 5.6 A curve C is said to be closed if it has the same start and end points.

In the case that C is a closed curve, we use the notation

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C \boldsymbol{F} \cdot d\boldsymbol{r}.$$

The circle over the integral sign does not mean anything extra- it's just a reminder that the path C we're integrating along is closed.

Let us write $-C$ to denote the curve with the same shape as C, but traveresed in the opposite direction. Then

$$\int_{-C} \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_{C} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

Observe that two curves C_1 and C_2 have the same start and end points precisely when we can join C_1 and $-C_2$ to obtain a closed curve. We use this to see that a path integral of a vector field $\boldsymbol{F}$ is independent of the curve, depending only on its endpoints, if and only if

$$\oint_V \boldsymbol{F} \cdot d\boldsymbol{r} = 0$$

for every closed curve C.

Theorem 5.7 (Green's Theorem) *Let R be a bounded region of the (x, y)-plane, and let C be the curve along the boundary of R, oriented in such a way that R is on the left as we advance along C (ie: we go anticlockwise).*

Let $\boldsymbol{F} = L(x, y)\boldsymbol{i} + M(x, y)\boldsymbol{j}$ be a two-dimensional vector field which is defined everywhere on R. Then

$$\int\int_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) \, dx \, dy = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

We also write this formula

$$\int\int_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) \, dx \, dy = \oint_C M \, dx + N \, dy$$

Proof: We will prove the result for the special case where R can be represented in both the forms

$$a \leq x \leq b \qquad u(x) \leq y \leq v(x)$$

and

$$c \leq y \leq d \qquad p(x) \leq x \leq q(y)$$

Then

$$\int\int_R \frac{\partial L}{\partial y} \, dx \, dy = \int_a^b \left(\int_{u(x)}^{v(x)} \frac{\partial L}{\partial y} \, dy \right) \, dx$$

Now

$$\int_{u(x)}^{v(x)} \frac{\partial L}{\partial y} \, dy = L(x, v(x)) - L(x, u(x))$$

so

$$\int\int_R \frac{\partial L}{\partial y} \, dx \, dy = \int_a^b L(x, v(x)) \, dx - \int_a^b L(x, u(x)) \, dx$$

The curve C is obtained by joining two curves C_1 and $-C_2$, where C_1 and C_2 have equations

$$\boldsymbol{r}(t) = t\boldsymbol{i} + u(t)\boldsymbol{j} \qquad a \leq t \leq b$$

and

$$\boldsymbol{r}(t) = t\boldsymbol{i} + v(t)\boldsymbol{j} \qquad a \leq t \leq b$$

respectively. Observe

$$\int_{C_1} L(x,y)\ dx = \int_a^b L(t,u(t))\ dt \qquad \int_{C_1} L(x,y)\ dx = \int_a^b L(t,v(t))\ dt$$

Therefore

$$\int\int_R \frac{\partial L}{\partial y}\ dx\ dy = -\oint_C L(x,y)\ dx$$

Similarly,

$$\int\int_R \frac{\partial M}{\partial x}\ dx\ dy = \oint_C M(x,y)\ dx$$

Putting these together

$$\int\int_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)\ dx\ dy = \oint_C M\ dx + N\ dy$$

□

Example 5.8 Use Green's theorem to evaluate the line integral

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

where C is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,2)$, and $\boldsymbol{F}(x,y) = (xy, x^2y^3)$.

Solution: We can write this line integral

$$L = \oint_C L\ dxM\ dy \qquad L(x,y) = xy,\ M(x,y) = x^2y^3$$

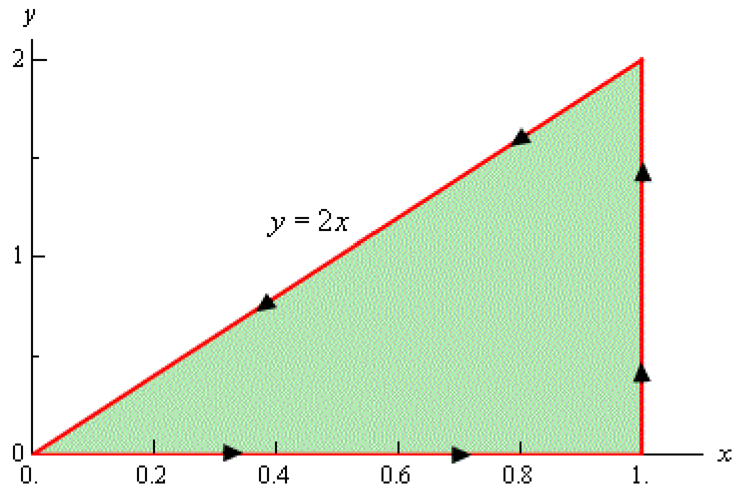
Then

$$\frac{\partial L}{\partial y} = x \qquad \frac{\partial M}{\partial x} = 2xy^3$$

so by Green's theorem

$$L = \int\int_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)\ dx\ dy = \int\int_R 2xy^3 - x\ dx\ dy$$

where R is the interior of the triangle. A drawing is now helpful.



We see it is easier to integrate with respect to y first, so

$$L = \int\int_R 2xy^3 - x \; dy \; dx = \int_0^1 \int_0^{2x} 2xy^3 - x \; dy \; dx = \int_0^1 \left[\frac{1}{2}xy^4 - xy\right]_0^{2x} \; dx$$

$$= \int_0^1 8x^5 - 2x^2 \; dx = \left[\frac{4}{3}x^6 - \frac{2}{3}x^2\right]_0^1 = \frac{2}{3}.$$

□

5.3 Volume Integrals

In the previous chapter, we saw how to evaluate integrals of functions of two variables over an area. We can do the same thing with functions of three variables over a three-dimensional volume. Rather than double integrals, these are triple integrals.

To be a little more precise, a *volume integral* is an integral of the form

$$\int\int\int_R f(x, y, z) \; dV$$

where R is a region in three-dimensional space, and $dV = dx \; dy \; dz$ is the *element of volume*. As in the two-dimensional case, at least for continuous functions we can change the order of integration, going first with whichever variable seems most appropriate.

Consider a function of two variables, $f(x, y)$ defined for $a \leq x \leq b$ and $c \leq y \leq d$. Then we can form the *repeated integral*

$$I = \int_c^d \int_a^b f(x, y) \; dx \; dy$$

This means integrate first with respect to x, treating y as a constant, to get a function of y, then with respect to y. In other words

$$I = \int_c^d \left(\int_a^b f(x,y)\ dx \right)\ dy$$

Example 5.9 Evaluate

$$I = \int \int \int_R x^2 + 4xy + z^2\ dV$$

where R is the region bounded by $x = 0$, $x = a$, $y = 0$, $y = b$ and $z = 0$, $z = c$.

Solution: We have

$$\begin{aligned} I = \int_0^c \int_0^b \int_0^a x^2 + 4xy + z^2\ dx\ dy\ dz &= \int_0^c \int_0^b \left[\frac{1}{3}x^3 + 2x^2y + z^2x \right]_0^a\ dy\ dz \\ = \int_0^c \int_0^b \frac{a^3}{3} + 2a^2y + z^a\ dy\ dz &= \int_0^c \left[\frac{a^3}{3}y + a^2y^2 + z^2ay \right]_0^b\ dz \\ = \int_0^c \frac{a^3}{3}b + a^2b^2 + z^2ab\ dz &= \left[\frac{a^3}{3}bz + a^2b^2z + \frac{1}{3}z^3ab \right]_0^c \\ = \frac{a^3}{3}bc + a^2b^2c + \frac{1}{3}c^3ab &= abc\left(\frac{a^2}{2} + ab + \frac{c^2}{3} \right) \end{aligned}$$

□

More sophisticated examples of volume integrals often involve cylindrical polar or spherical polar coordinates.

Recall that in *cylindrical polar coordinates*, a point P with position vector (x, y, z) is represented by three coordinates r, θ and z where $r \geq 0$, $0 \leq \theta \leq 2\pi$, and

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z \text{ as before}$$

The following is similar to computing the element of area in plane polar coordinates.

Proposition 5.10 *In cylindrical polar coordinates, the element of volume is defined by*

$$dV = r\ dr\ d\theta\ dz$$

□

Another choice of three-dimensional coordinate system that is, as we have seen, useful for some problems, are *spherical polar coordinates*. Here a point P with position vector (x, y, z) is represented by three coordinates r, θ and ϕ where $r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

Proposition 5.11 *In spherical polar coordinates, the element of volume is defined by*

$$dV = r^2 \sin\theta \; dr \; d\theta \; d\phi$$

□

Example 5.12 The hemisphere bounded by $x^2+y^2+z^2 = a^2$, $z \geq 0$ is occupied by a material with density

$$\rho = \rho_0 \frac{z}{a}$$

Find the mass of the material.

Solution: The total mass is the integral

$$M = \int\int\int_R \rho \; dV = \int\int\int_R \rho_0 \frac{z}{a} \; dV$$

In spherical polar coordinates (r, θ, ϕ), we have $z = r\cos\theta$, and the hemisphere is defined by saying $0 \leq r \leq a$, $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi \leq 2\pi$. We also have $dV = r^2 \sin\theta \; dr \; d\theta \; d\phi$, so

$$M = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^a \frac{\rho_0}{a} r\cos\theta \, r^2 \sin\theta \; dr \; d\theta \; d\phi = \frac{\rho_0}{a} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^a r^3 \sin\theta\cos\theta \; dr \; d\theta \; d\phi$$

$$= \frac{\rho_0}{a} \left[\frac{r^4}{4}\right]_0^a \left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} [\phi]_0^{2\pi} = \frac{\pi a^3}{4}\rho_0$$

□

5.4 Surface Integrals

A smooth curve is a smooth function $\boldsymbol{r}(t)$ of one variable. A *smooth surface*, S, can be viewed as a vector valued function of two variables, $\boldsymbol{r}(u, v)$. The function $\boldsymbol{r}(u, v)$ is said to be the function that *parametrises* S.

Surfaces are often defined by equations of the form $f(x, y, z) = 0$ or $z = g(x, y)$. In this case, to calculate surface integrals, we need to find a parametrisation.

Example 5.13 The surface $z = g(x, y)$ can be represented parametrically by the function

$$\boldsymbol{r}(x, y) = x\boldsymbol{i} + y\boldsymbol{j} + g(z, y)\boldsymbol{k}.$$

Example 5.14 The sphere $x^2+y^2+z^2 = a^2$ can be written in terms of spherical polar coordinates $(r.\theta, \phi)$ by $r = a$. Since

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

this gives us a parametrisation

$$\boldsymbol{r}(\theta, \phi) = a \sin\theta \cos\phi \boldsymbol{i} + a \sin\theta \sin\phi \boldsymbol{j} + a \cos\theta \boldsymbol{k}$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$.

Let us assume that for a surface with parametrisation $\boldsymbol{r}(u, v)$, we have

$$\boldsymbol{N} = \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \neq \mathbf{0}.$$

Then $\boldsymbol{N}$ is a normal vector to S, and we have a *unit normal*

$$\boldsymbol{n} = \frac{\boldsymbol{N}}{|\boldsymbol{N}|}.$$

As a matter of convention, in a closed surface such as a sphere, we arrange matters so that the unit normal points outwards.

Definition 5.15 Let $\boldsymbol{F}$ be a vector field, and let S be a smooth surface, parametrised by a function $\boldsymbol{r}(u, v)$, with (u, v) in a region R. Then we define the *surface integral* of $\boldsymbol{F}$ on S by the equation

$$\int\int_S \boldsymbol{F} \cdot \boldsymbol{n} \; dA = \int\int_R \boldsymbol{F}(\boldsymbol{r}(u, v)) \cdot \boldsymbol{N} \; du \; dv.$$

The element of area dA is given by the formula

$$\boldsymbol{n} \; dA = \boldsymbol{N} \; du \; dv \qquad \boldsymbol{N} = \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}$$

Example 5.16 In spherical polar coordinates (r, θ, ϕ) the vector field $\boldsymbol{F}$ is defined to be

$$\boldsymbol{F} = r^2 \hat{\boldsymbol{r}} + r \cos\theta \hat{\boldsymbol{\theta}}$$

where $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ are unit vectors in the direction of increasing r and increasing θ.

Evaluate

$$\int\int_S \boldsymbol{F} \cdot \boldsymbol{n} \; dS$$

where S is the sphere $r = a$.

Solution: The sphere is given parametrically by

$$\boldsymbol{r}(\theta, \phi) = a \sin\theta \cos\phi \boldsymbol{i} + a \sin\theta \sin\phi \boldsymbol{j} + a \cos\theta \boldsymbol{k}$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$.

Observe

$$\frac{\partial \boldsymbol{r}}{\partial \theta} = a \cos\theta \cos\phi \boldsymbol{i} + a \cos\theta \sin\phi \boldsymbol{j} - a \sin\theta \boldsymbol{k} \qquad \frac{\partial \boldsymbol{r}}{\partial \phi} = -a \sin\theta \sin\phi \boldsymbol{i} + a \sin\theta \cos\phi \boldsymbol{j}$$

so

$$\boldsymbol{N} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a\cos\theta\cos\phi & a\cos\theta\sin\phi & -a\sin\theta \\ -a\sin\theta\sin\phi & a\sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= a^2\sin^2\theta\cos\phi\boldsymbol{i} + a^2\sin^2\theta\cos\phi\boldsymbol{j} + a^2\cos\theta\cos\phi(\cos^2\phi + \sin^2\phi)\boldsymbol{k}$$

Now

$$\hat{\boldsymbol{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

and $\cos^2\phi + \sin^2\phi = 1$.

We see that

$$\boldsymbol{n}\ dA = \boldsymbol{N}\ d\theta\ d\phi = a^2\sin\theta\hat{\boldsymbol{r}}\ d\theta\ d\phi.$$

Now

$$\begin{aligned} \int\int_S \boldsymbol{F}\cdot\boldsymbol{n}\ dS &= \int_0^{2\pi}\int_0^{\pi}(a^2\hat{\boldsymbol{r}} + a\cos\theta\hat{\boldsymbol{\theta}})\cdot a^2\sin^2\theta\hat{\boldsymbol{r}}\ d\theta\ d\phi \\ &= \int_0^{2\pi}\int_0^{\pi} a^4\sin\theta\ d\theta d\phi \\ &= a^4\left[\phi\right]_0^{2\pi}\left[-\cos\theta\right]_0^{\pi} = 4\pi a^4. \end{aligned}$$

□

Surface integrals have an important physical significance in areas such as fluid dynamics and electromagnetism.

To give an instance in fluid dynamics, if a region inside a closed surface S contains a fluid with density ρ, moving with velocity vector field $\boldsymbol{v}$, then we say that the *flux* of fluid in the direction of a unit vector $\boldsymbol{u}$ is $\rho\boldsymbol{v}\cdot\boldsymbol{u}$.

The flux is the mass of fluid crossing a unit area with normal vector $\boldsymbol{u}$ in unit time. Thus, per unit the amount of fluid leaving the surface S at an element of area dS is $\rho\boldsymbol{v}\cdot\boldsymbol{n}dS$.

In other words, the rate at which the fluid leaves S (in terms of mass per unit time) can be computed in terms of the flux as the surface integral

$$\int\int_S \rho\boldsymbol{v}\cdot\boldsymbol{n}\ dS.$$

5.5 Gauss' Divergence Theorem

Recall that Green's theorem relates a line integral of a function around a closed curve with the area integral of a related function over a region bounded by the curve.

Gauss' divergence theorem is a three-dimensional version of the same notion, relating the surface integral of a function on a closed surface with the volume integral of a related function over the volume bounded by the surface. Specifically, we have the following.

Theorem 5.17 *Let S be a closed surface in three-dimensional space, with outward-pointing unit normal $\boldsymbol{n}$. Let R be the volume bounded by S. Let vF be a vector field. Then*

$$\int\int\int_R \boldsymbol{\nabla}\cdot\boldsymbol{F}\ dV = \int\int_S \boldsymbol{F}\cdot\boldsymbol{n}\ dA$$

□

We saw previously, that in a fluid with velocity vector field $\boldsymbol{v}$. the amount per unit time leaving a volume R with surface S is

$$\int\int_S \rho\boldsymbol{v}\cdot\boldsymbol{n}\ dS.$$

By the above, this quantity is the same as that in the integral

$$\int\int\int_R \boldsymbol{\nabla}\cdot\boldsymbol{v}\ dV$$

Example 5.18 Let S be the surface consisting of the cylinder $x^2+y^2=a^2$, for $0\le z\le b$, and the discs $z=0$ and $z=b$ (with $0\le x^2+y^2\le a^2$ at these ends). Let

$$\boldsymbol{F}=(x^3,x^2y,x^2z)$$

Use Gauss' divergence theorem to evaluate the surface integral

$$I=\int\int_S \boldsymbol{F}\cdot\boldsymbol{n}\ dA$$

Solution: The surface S bounds the cylinder, R, given by the inequalities $x^2+y^2\le a^2$, $0\le z\le b$. We have

$$\boldsymbol{\nabla}\cdot\boldsymbol{F}=3x^2+x^2+x^2=5x^2$$

so by Gauss' divergence theorem

$$I=\int\int\int_R 5x^2\ dV$$

Now, the shape of R means it makes sense to use cylindrical polar coordinates (r,θ,z), where $x=r\cos\theta$, $y=r\sin\theta$, and z is as before. The region r is defined by the inequalities $0\le r\le a$, $0\le z\le b$, and $0\le\theta\le 2\pi$. The element of volume is given by

$$dV=r\ dr\ d\theta\ dz$$

Hence

$$I=\int_0^b\int_0^{2\pi}\int_0^a 5r^2\cos^2\theta\ r\ dr\ d\theta\ dz=5\int_0^b\int_0^{2\pi}\int_0^a r^3\cos^2\theta\ r\ dr\ d\theta\ dz$$

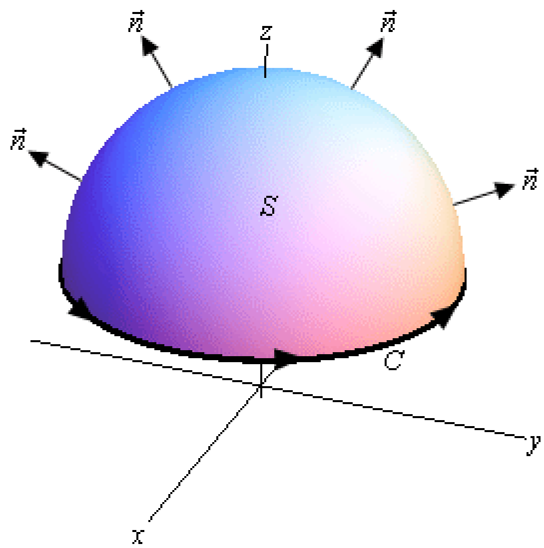
$$=5\left[z\right]_0^b\left[\frac{1}{2}(1+\cos 2\theta)\right]_0^{2\pi}\left[\frac{1}{4}r^4\right]_0^a=\frac{5}{4}\pi ba^4$$

□

5.6 Stokes' Theorem

Green's theorem relates a line integral of a function around a closed curve in the plane with the area integral of a related function over a region bounded by the curve.

In three dimensional space, a surface S may have a curve C with a boundary, as shown.



Let us parametrise S so that the curve is traversed in such a way that the direction and the unit normal of S are related by the *right-hand rule*, that is if the fingers of the right hand give the direction along C, the thumb points in the direction of $\boldsymbol{n}$. Then the following holds.

Theorem 5.19 (Stokes' theorem) *Let S and C be as above. Let $\boldsymbol{F}$ be a vector field. Then*

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int\int_S (\boldsymbol{\nabla} \times \boldsymbol{F}.\boldsymbol{n} \; dS$$

□

If the surface is contained in the plane, then this result becomes Green's theorem, which we have already seen. Stokes' theorem also gives us a convenient way to prove the following result stated back in chapter 3.

Theorem 5.20 *Let $\boldsymbol{F}$ be a vector field where $\boldsymbol{\nabla} \times \boldsymbol{F} = \times 0$. Then $\boldsymbol{F} = \boldsymbol{\nabla}\phi$ for some scalar field ϕ.*

Proof: Let C be a closed curve. Then by Stokes' theorem

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int \int_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot \boldsymbol{n} \, dS = 0$$

as $\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$.

Thus, a line integral of $\boldsymbol{F}$ is independent of the curve, depending only on its endpoints. But this in turn means, by theorem 5.5 that $\boldsymbol{F} = \boldsymbol{\nabla}\phi$ for some ϕ. □