Data provided: Formula sheet

MAS165

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012-2013**

Mathematics for Physicists **2 hours**

You should attempt ALL questions of this exam.

Section A

A1 A plane is given by the equation

$$4x + 5y + 7z = 21$$

and a line by the equation $\mathbf{r} = (1, 2, 3) + \lambda(1, 2, -2)$, where λ is a real parameter.

(i) Show that the line does not intersect the plane. ***(4 marks)***

(ii) Therefore, calculate the distance of the line to the plane. ***(4 marks)***

(iii) Find the direction of the line of intersection of the two planes $x+3y-z = 5$ and $2(x - y) + 4z = 3$. ***(5 marks)***

A2 Show that $f(x, y) = e^{-x}\cos(y) - e^{-y}\cos(x)$ obeys the two dimensional Laplace equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \ .$$

Show that the function $g(x, y) = \sqrt{x^2 + y} - xy$ does not obey the two dimensional Laplace equation. ***(9 marks)***

A3 Stokes' theorem may be written:

$$\oint_C \mathbf{G}.\mathbf{dr} = \int_S (\nabla \times \mathbf{G}).\hat{\mathbf{n}}\, dS$$

Indicate whether the following statements about Stokes' theorem, as expressed here, are true or false

(i) The term $(\nabla \times \mathbf{G})$ is the divergence of the vector field $\mathbf{G}$.

(ii) $\hat{\mathbf{n}}$ is a unit vector parallel with the boundary C.

(iii) $\int_S dS$ is a surface integral, over the surface S.

(3 marks)

Section B

B1 (i) Find the work done by a force $\mathbf{F} = (x + yz)\mathbf{i} + (y + xz)\mathbf{j} + (z + xy)\mathbf{k}$ in moving a particle from the origin $\mathcal{O}$ to the point $A(1, 1, 1)$

(a) along the curve $x = t, y = t^2, z = t^3$,

(b) along the straight line $\mathcal{O}A$. ***(12 marks)***

(ii) A scalar function is given as

$$\phi(x, y, z) = x^2 - y \sin(x - z).$$

(a) Calculate the gradient of $\phi(x, y, z)$, i.e. calculate $\mathbf{V} = \nabla\phi$. ***(3 marks)***

(b) Using your result, calculate the divergence of $\mathbf{V}$. ***(4 marks)***

(c) By explicit calculation, show that $\nabla \times \mathbf{V} = 0$. ***(6 marks)***

B2 (i) A vector field is given by

$$\mathbf{V} = V_1\hat{\mathbf{r}} + V_2\hat{\boldsymbol{\theta}} + V_3\hat{\mathbf{z}} = r\hat{\mathbf{r}} + (a + r^3)\hat{\boldsymbol{\theta}} + b\ln(z)\hat{\mathbf{z}}$$

in cylindrical polar coordinates, where a and b are positive constants. Calculate the divergence and curl of the vector field, given that the divergence and curl may be expressed in cylindrical coordinates as

$$\nabla.\mathbf{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_1) + \frac{1}{r}\frac{\partial}{\partial \theta}(V_2) + \frac{\partial}{\partial z}(V_3)$$

and

$$\nabla \times \mathbf{V} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ V_1 & rV_2 & V_3 \end{vmatrix}$$

respectively. Can $\nabla \times \mathbf{V}$ be zero? ***(10 marks)***

(ii) Sketch the region of integration represented by the repeated integral

$$\int\int_R x^2 y \, dx \, dy$$

where R is the region such that $x \geq 0$, $y \geq 0$, and $x^2 + y^2 \leq a^2$. By transforming to plane polar coordinates, evaluate the integral.

(15 marks)

B3 (i) Verify the divergence theorem

$$\int_V (\nabla \cdot \mathbf{A}) \, dV = \oint_S \mathbf{A} \cdot \hat{\mathbf{n}} \, dS \, ,$$

for the vector field $\mathbf{A} = (x, y, z)$ and S being the surface enclosing a cylinder of radius a (i.e. $x^2 + y^2 = a^2$) and height h. The bottom surface of the cylinder lies in the $xy-$plane ($z = 0$). Hint: split the surface integral into three parts. To find $\hat{\mathbf{n}}$ for the curved surface of the cylinder, note that you can find it by calculating $\nabla\phi$, with $\phi(x, y) = x^2 + y^2 = a^2$ describing the surface. ***(15 marks)***

(ii) A magnetic field is given, in cylindrical polar coordinates (r, θ, z), as $\mathbf{H} = H_0 r^2 \hat{\boldsymbol{\theta}}/a^2$, with $r \leq a$, where H_0 and a are positive constants. The magnetic field vanishes for $r > a$. Evaluate

$$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{x},$$

where $\mathcal{C}$ is the circle $z = 0$, $r = R$, described in the anticlockwise sense for $R < a$.

(10 marks)

End of Question Paper