MAS165 - 2012/2013

A1

(i) For line & plane to intersect we need to have $\vec{n} \cdot \vec{a} \neq 0$, where $\vec{n}$ is $\perp$ to the plane and $\vec{a}$ is the direction of the line. 1

$4x + 5y + 7z = 21 \quad \Rightarrow \vec{n} = (4,5,7)$ 1

$\vec{r} = (1,2,3) + \lambda(1,2,-2) \Rightarrow \vec{a} = (1,2,-2)$ 1

$\vec{n} \cdot \vec{a} = 4 + 10 - 14 = 0$

$\Rightarrow$ Line cannot intersect plane. 1 $\boxed{4}$

(ii) $(0,0,3)$ is point on plane. 1

$\vec{b} = (1,2,3) - (0,0,3) = (1,2,0)$ is a vector pointing from a point on plane to a point on the line. Therefore, 1

$$d = \left| (1,2,0) \cdot \frac{\vec{n}}{|\vec{n}|} \right| = \frac{1}{\sqrt{90}} (4+10) = \frac{14}{\sqrt{90}} \quad 2$$

is the distance of line to plane. $\boxed{4}$

(iii)

Direction of intersection is

$\vec{n}_1 \times \vec{n}_2$ (or $-\vec{n}_1 \times \vec{n}_2$). (2)

$\vec{n}_1 = (1,3,-1)$, $\vec{n}_2 = (2,-2,4)$ (2)

$\Rightarrow \vec{n}_1 \times \vec{n}_2$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -1 \\ 2 & -2 & 4 \end{vmatrix} = (10, -6, -8)$$ (1)

$\boxed{5}$

A2 $f(x,y) = e^{-x}\cos y - e^{-y}\cos x$

$\Rightarrow \dfrac{\partial f}{\partial x} = -e^{-x}\cos y + e^{-y}\sin x$ (1)

$\dfrac{\partial^2 f}{\partial x^2} = e^{-x}\cos y + e^{-y}\cos x$ (1)

$\dfrac{\partial f}{\partial y} = e^{-x}(-\sin y) + e^{-y}\cos x$ (1)

$\dfrac{\partial^2 f}{\partial y^2} = -e^{-x}\cos y - e^{-y}\cos x$ (1)

$\Rightarrow \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = e^{-x}\cos y + e^{-y}\cos x$

$- e^{-x}\cos y - e^{-y}\cos x$

$= 0$ (1)

(3)

$$g(x,y) = \sqrt{x^2+y} - xy$$

$$\frac{\partial g}{\partial x} = \frac{x}{\sqrt{x^2+y}} - y \quad (1) \quad , \quad \frac{\partial^2 g}{\partial x^2} = \frac{1}{\sqrt{x^2+y}} - \frac{x^2}{(x^2+y)^{3/2}} \quad (1)$$

$$\frac{\partial g}{\partial y} = \frac{1}{2\sqrt{x^2+y}} - x \quad (1) \quad , \quad \frac{\partial^2 g}{\partial y^2} = \frac{-1}{\cancel{2}4\sqrt{x^2+y}^{\,3}} \quad (1)$$

$$\Rightarrow \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \neq 0$$

$\boxed{9}$

A3

(i) F ($\vec{\nabla} \times \vec{G}$ is curl!)

(ii) F ($\hat{\vec{n}}$ is $\perp$ to surface)

(iii) T

$\boxed{3}$

B1

i) $W = \int_{O}^{A} \vec{F}\cdot d\vec{r}$ 1

a) $d\vec{r} = (1, 2t, 3t^2)\,dt$ 1

Since $\vec{r} = (t, t^2, t^3)$, t runs from 0 to 1. 1

$\vec{F}\cdot d\vec{r} = (x+yz)\,dx + (y+xz)\,dy + (z+xy)\,dz$

$= (t+t^5)\,dt + (t^2+t^4)\,2t\,dt + (t^3+t^3)\,3t^2dt$

$= dt\left[t + t^5 + 2t^3 + 2t^5 + 3t^5 + 3t^5\right]$

$= dt\left[t + 9t^5 + 2t^3\right]$ 2

$\Rightarrow W = \int_0^1 (t + 9t^5 + 2t^3)\,dt$ 1

1

$= \left[\frac{1}{2}t^2 + \frac{9}{6}t^6 + \frac{1}{2}t^4\right]_0^1 = \frac{1}{2} + \frac{9}{6} + \frac{1}{2}$

$= \frac{15}{6} = \underline{\underline{\frac{5}{2}}}$ 1

1

b) Straight line $\vec{r} = (1,1,1)\,t$, $t = 0 \ldots 1$

$\Rightarrow \vec{F}\cdot d\vec{r} = (t+t^2)\,dt + (t+t^2)\,dt + (t+t^2)\,dt$

~~statt~~ $(3t + 3t^2)\,dt$ 1

$\Rightarrow W = \int_0^1 (3t + 3t^2)\,dt = \frac{3}{2} + 1 = \underline{\underline{\frac{5}{2}}}$

1

1

$\boxed{12}$

(5)

(ii) a)

$$\vec{V} = \vec{\nabla}\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

$$= \left(2x - y\cos(x-z), -\sin(x-z), +y\cos(x-z)\right)$$

(1) (1) (1)

$\boxed{3}$

b)

$$\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \quad (1)$$

$$= 2 + y\sin(x-z) + 0 + y\sin(x-z) \quad (2)$$

$$= 2(1 + y\sin(x-z)) \quad (1)$$

$\boxed{4}$

c)

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \vec{i}\left(\cos(x-z) + (-\cos(x-z)\right) \quad (2)$$

$$+ \vec{j}\left(y\sin(x-z) - y\sin(x-z)\right) \quad (2)$$

$$+ \vec{k}\left(-\cos(x-z) + \cos(x-z)\right)_{(2)} = (0, 0, 0)$$

$\boxed{6}$

B2

(i) $\vec{\nabla} \cdot \vec{V} = \frac{1}{r}\left(\frac{\partial(rV_1)}{\partial r}\right) + \frac{1}{r}\frac{\partial V_2}{\partial \theta} + \frac{\partial V_3}{\partial z}$

$= \frac{1}{r}\frac{\partial r^2}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta}(a + r^3) + \frac{\partial}{\partial z}(b \ln z)$ (2)

$= 2 + 0 + \frac{b}{z} = 2 + \frac{b}{z}$ (2)

$\vec{\nabla} \times \vec{V} = \frac{1}{r}\Big\{ \hat{\vec{r}}\left(\frac{\partial V_3}{\partial \theta} - \frac{\partial(rV_2)}{\partial z}\right)$

$- r\hat{\vec{\theta}}\left(\frac{\partial V_3}{\partial r} - \frac{\partial V_1}{\partial z}\right)$

$+ \hat{\vec{z}}\left(\frac{\partial(rV_2)}{\partial r} - \frac{\partial V_1}{\partial \theta}\right)\Big\}$ (3)

$= \frac{1}{r}\Big(\hat{\vec{r}}(0-0) - r\hat{\vec{\theta}}(0-0)$

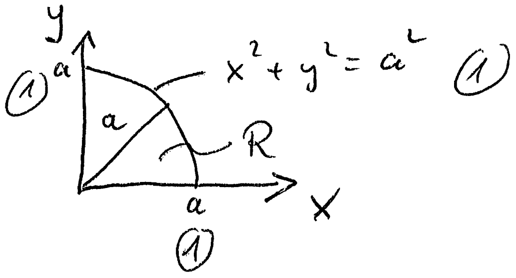
$+ \hat{\vec{z}}\left(V_2 + r3r^2 - 0\right)\Big)$

$= \frac{1}{r}\hat{\vec{z}}\left(a + r^3 + 3r^3\right) = \frac{a + 4r^3}{r}\hat{\vec{z}}$ (2)

Since $a > 0$, the curl does not vanish. (1)

$\boxed{10}$

(ii)



$x = r\cos\phi$ (1), $y = r\sin\phi$ (1), $dx\,dy = r\,dr\,d\phi$ (1)

$$I = \iint_R x^2 y \, dx\,dy = \iint_R r^2\cos^2\phi \; r\sin\phi \; r\,dr\,d\phi$$

$$= \int_{\phi=0}^{\pi/2} \int_{r=0}^{a} r^4\,dr \; \cos^2\phi \sin\phi \, d\phi$$

(1) (1) (1) (1)

$$= \left[\frac{1}{5}r^5\right]_0^a \times \underbrace{\int_0^{\pi/2} \cos^2\phi \sin\phi \, d\phi}_{y} = \frac{1}{5}a^5 \times y$$

(1)

Now $\frac{d\cos\phi}{d\phi} = -\sin\phi \rightarrow \sin\phi \, d\phi = -d(\cos\phi)$ (1)

$$\Rightarrow y = \int_{(0)}^{(\pi/2)} \cos^2\phi \sin\phi \, d\phi = -\int_{(0)}^{\pi/2} u^2\,du = -\frac{1}{3}u^3\Big|_{(0)}^{\pi/2}$$ (1)

$$= -\frac{1}{3}\cos^3\phi\Big|_0^{\pi/2} = -\frac{1}{3}\left(\underset{\uparrow 0}{\cos^3\frac{\pi}{2}} - \underset{\uparrow 1}{\cos^3 0}\right) = +\frac{1}{3}$$ (1)

$$\Rightarrow \underline{\underline{I = \frac{1}{15}a^5}}$$ (1)

$\boxed{15}$

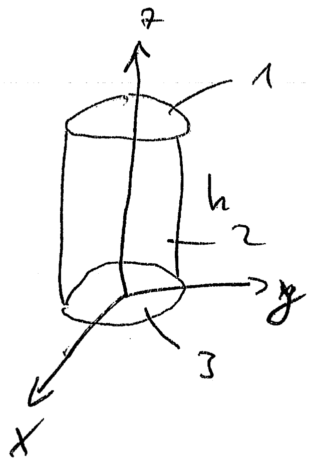
B3

(i) $\vec{A} = (x, y, z) \Rightarrow \vec{\nabla} \cdot \vec{A} = 3$ (1)

$$\iiint_V dV\, \vec{\nabla} \cdot \vec{A} = 3 \iiint_V dV \text{, with } V = \text{volume of cylinder.}$$ (1)

$$\Rightarrow \iiint_V dV\, \vec{\nabla} \cdot \vec{A} = 3 \cdot V = 3\pi a^2 h$$ (1)

This should be equal to $\oint \vec{A} \cdot \hat{\vec{n}}\, dS$



For surface 1: $\hat{\vec{n}} = \vec{k}$ (1)

For surface 3: $\hat{\vec{n}} = -\vec{k}$ (1)

$$1: \iint_{S_1} \vec{A} \cdot \hat{n}\, dS = \iint_{S_1} z\, dx\, dy$$

$$= h \iint_{S_1} dx\, dy = h\pi a^2$$ (1)

$$3: \iint_{S_3} \vec{A} \cdot \hat{n}\, dS = -\iint_{S_3} z\, dx\, dy = 0 \quad \text{since } z = 0 \text{ on } S_3$$ (1)

To find $\hat{\vec{n}}$ on S_2, note that $\hat{\vec{n}} \propto \vec{\nabla}\phi$ with $\phi = x^2 + y^2 - a^2$

$$\Rightarrow \hat{\vec{n}} = \frac{(2x, 2y, 0)}{\sqrt{4x^2 + 4y^2}} = \frac{(x, y, 0)}{\sqrt{x^2 + y^2}} = \frac{(x, y, 0)}{a}$$ (2)

$$\vec{A} \cdot \hat{\vec{n}} = \frac{1}{a}(x, y, z) \cdot (x, y, 0) = \frac{1}{a}(x^2 + y^2 + 0) = a,$$ (2)

since $x^2 + y^2 = a^2$ on S_2.

$$\Rightarrow \iint_{S_2} \vec{A}\cdot\hat{\vec{n}}\, dS = a \iint_{S_2} dS = a \cdot S_2 \quad (1)$$

$$= 2\pi a^2 h \quad (1)$$

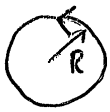
$\Rightarrow$ In total $\quad 2\pi a^2 h + \pi h a^2 = 3\pi a^2 h \quad (1)$

$$\Rightarrow \iiint_V dV\, \vec{\nabla}\cdot\vec{A} = \oint_S \vec{A}\cdot\hat{\vec{n}}\, dS \text{ , as it should.} \quad (1)$$

$\boxed{15}$

(ii) $\vec{H} = H_0 \left(\frac{r}{a}\right)^2 \hat{\vec{\Theta}}$ with $r \le a$.

We integrate over circle:



(1)

On the circle $d\vec{r} = R\, d\theta\, \hat{\vec{\Theta}}$ (3), so $\quad$ (since $R < a$)

$$\oint \vec{H}\cdot d\vec{r} = \int_{\theta=0\,(1)}^{2\pi\,(1)} H_0 \frac{R^2}{a^2}\,(1)\, R\, \underbrace{\hat{\vec{\Theta}}\cdot\hat{\vec{\Theta}}}_{=1\,(1)}\, d\theta$$

$$= \int_{\theta=0}^{2\pi} H_0 \frac{R^3}{a^2}\, d\theta = \frac{H_0 R^3}{a^2} \left[\theta\right]_0^{2\pi} \quad (1)$$

$$= 2\pi \frac{H_0 R^3}{a^2} \quad (1)$$

$\boxed{10}$