



The
University
Of
Sheffield.

MAS165

SCHOOL OF MATHEMATICS AND STATISTICS

**Spring Semester
2011-2012**

Mathematics for Physicists

2 hours

This paper comprises TWO sections. You should attempt ALL questions on this paper.

Section A

- A1** Given the point $A = (1, 2, -1)$ and the line $\mathbf{r} = (1, 0, 1) + \lambda(3, 0, -1)$.
- (i) Find the shortest distance from A to the line. *(5 marks)*
 - (ii) Find the equation of the plane passing through A whose normal is parallel to \mathbf{r} . *(4 marks)*
 - (iii) Find the perpendicular distance of the plane from the origin $(0,0,0)$. *(3 marks)*
- A2** (i) Show that $(\mathbf{c} \times (\mathbf{b} \times \mathbf{c})) \times \mathbf{c} = c^2 \mathbf{b} \times \mathbf{c}$, where c is the magnitude of the vector \mathbf{c} . *(3 marks)*
- (ii) If $\mathbf{r} = (1, 1, 1)$, $\mathbf{s} = (2, 0, \lambda)$ and $\mathbf{t} = (0, 1, 3)$, find $\mathbf{r} \times \mathbf{s}$ and $\mathbf{s} \times \mathbf{t}$ and hence show that

$$\mathbf{r} \cdot (\mathbf{s} \times \mathbf{t}) = (\mathbf{r} \times \mathbf{s}) \cdot \mathbf{t}$$

is true for all values of λ . *(6 marks)*

A3 Gauss' theorem may be written:

$$\oint_S \mathbf{G} \cdot \hat{\mathbf{n}} dS = \int_V (\nabla \cdot \mathbf{G}) dV .$$

Indicate whether the following statements about Gauss' theorem, as expressed here, are true or false

- (i) The term $(\nabla \cdot \mathbf{G})$ is the curl of the vector field \mathbf{G} .
- (ii) The integral on the right hand side is a volume integral.
- (iii) The theorem relates the behaviour of the vector field \mathbf{G} inside the volume V to the behaviour of \mathbf{G} in the surface S that bounds V .
- (iv) $\oint_S dS$ is a surface integral, over the surface S .

(4 marks)

Section B

B1 (i) The electric potential $V(x, y, z)$ for a point charge Q situated at the origin may be expressed in appropriate units as

$$V = \frac{Q}{4\pi\epsilon_0\sqrt{x^2 + y^2 + z^2}},$$

where ϵ_0 is a positive constant. Verify that V satisfies Laplace's equation.

(10 marks)

(ii) The parametric equation of a right circular helix may be described as

$$\mathbf{r} = (a \cos \theta, a \sin \theta, b \theta),$$

where a and b are positive constants. Find the unit vector along the tangent to the curve.

(4 marks)

(iii) Evaluate the integral

$$I = \int \mathbf{r} \cdot d\mathbf{r},$$

where $\mathbf{r} = (a \cos \theta, a \sin \theta, b \theta)$, a and b are positive constants and $0 \leq \theta \leq 4\pi$.

(11 marks)

- B2** (i) The surface density of a plane lamina is given by

$$\sigma(x, y) = \sigma_0 \frac{x^2 + y^3}{a},$$

where σ_0 and a are constant. Find the mass of the lamina by evaluating the integral

$$M = \int_{\mathcal{R}} \sigma(x, y) dx dy,$$

where \mathcal{R} is the region bounded by the y -axis (i.e. $x = 0$), the line $y = 2a$ and the line $y = x$. Integrate first over x and then over y . Evaluate then the same integral again by reversing the order of integration (i.e. first integrating over y and then over x). **(15 marks)**

- (ii) Consider a surface specified by the equation

$$x^3 + y^3 + z^3 = 3.$$

Find the unit normal vector to the surface at the point $(1, 1, 1)$. Find furthermore the equation for the tangent plane at that point. **(5 marks)**

- (iii) Let $\phi(x, y, z)$ be a generic scalar function. Show by explicit calculation that the gradient of ϕ obeys

$$\nabla \times (\nabla \phi) = 0.$$

(5 marks)

- B3** (i) A scalar function is given in spherical polar coordinates by

$$V = \frac{A}{r} \sin^2 \theta \sin \phi,$$

where A is a positive number. Calculate the gradient of V . **(5 marks)**

- (ii) Given that

$$\frac{\partial f}{\partial x} = 3x^2 y^2 + y \cos x$$

and

$$\frac{\partial f}{\partial y} = 2x^3 y - 2y \sin(y^2) + e^y + \sin x,$$

find $f(x, y)$. **(7 marks)**

- (iii) Find the work done in a moving particle P along a curve C specified by

$$\mathbf{r} = (1, 2, -1) + (t, t^2, 2t), 0 \leq t \leq 1$$

against a force specified by $\mathbf{F} = 2xy \mathbf{i} + (x+y)\mathbf{k}$. **(13 marks)**

End of Question Paper