

# Categories of Unbounded Operators

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## Abstract

In this article we introduce the concept of an  $LK^*$ -algebroid, which is defined axiomatically. The main example of an  $LK^*$ -algebroid is the category of all subspaces of a Hilbert space and closed (not necessarily bounded) linear operators. We prove that for any  $LK^*$ -algebroid there is a faithful functor that respects its structure and maps it into this main example.

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## 1 Introduction

One of the nicest things about  $C^*$ -algebras, known since they were introduced (see [12]) is that any  $C^*$ -algebra can be represented as an algebra of bounded linear operators acting on a Hilbert space. In the classical algebraic approach to quantum field theory and statistical mechanics, as described in [11, 1]  $C^*$ -algebras are typically used to describe the algebra of observables and states, free from any particular model, and the deep mathematical theory of  $C^*$ -algebras (see for instance the books [8, 9, 10, 20]) is there to exploit.

However, as pointed out for instance in the introduction to [2], there is in principle a problem with the  $C^*$ -algebra approach. Specifically,  $C^*$ -algebras correspond to *bounded* linear operators on a Hilbert space, and most observables appearing in physical systems correspond to *unbounded* operators, such as differentiation, on a Hilbert space. One issue with handling unbounded operators

algebraically is that a well-behaved unbounded linear operator does not have a domain equal to the whole subspace it acts on, but rather a dense subset.

This issue led to the introduction of partial  $*$ -algebras and various elaborations such as quasi- $*$ -algebras,  $O^*$ -algebras and  $CQ^*$ -algebras; see for instance [2, 6, 5, 7]. One major area of interest in the study of these structures is the representation of one described axiomatically as a concrete partial algebra of not necessarily bounded operators on a Hilbert space.

As is the case for  $C^*$ -algebras, a partial  $*$ -algebra with suitable additional structure can always be represented in this way. Indeed, the proof of this largely follows the classical GNS construction for  $C^*$ -algebras (see [12]), but the partial multiplication means that the states used to construct the Hilbert space are replaced by slightly more fiddly constructions called *biweights* (see [4, 3, 22]) which explicitly involve sesquilinear forms and behave slightly awkwardly for homomorphisms.

In this article we present an alternative construction. Instead of considering partial  $*$ -algebras, we remember the domains of unbounded operators on a Hilbert space and consider *algebroids*. Doing this carefully, and describing a fair amount of extra structure, allows us to come up with a representation theory for unbounded operators which again uses states rather than biweights. The constructions are based on work for  $C^*$ -categories in [13, 19].

More specifically, we focus on algebroids where the morphism sets are locally convex vector spaces equipped with an  $A$ -bimodule structure, where  $A$  is a  $C^*$ -algebra, the objects are arranged in a lattice (inspired by looking at subspaces of a Hilbert space), a partially defined involution, and the condition that morphisms of the form  $xx^*$ , where  $x \mapsto x^*$ , have a positive spectrum. We call algebroids with this additional structure *LK\*-algebroid*. Precise details are in the next section.

We then look at examples of *LK\*-algebroids*. The primary example is the algebroid where the objects are all subspaces of a Hilbert space  $H$ , and the set of morphisms from a subspace  $U$  to a subspace  $V$  consists of all operators with domain  $U$  and image contained in  $V$ . We conclude by adapting the GNS construction to prove that for any *LK\*-algebroid* there is a faithful functor that respects its structure into the category of subspaces and operators on a Hilbert space.

## 2 Locally Convex Algebroids and Further Structures

In a small category  $\mathcal{C}$ , let us write  $Ob(\mathcal{C})$  to denote the set of objects,  $Hom(U, V)_{\mathcal{C}}$  to denote the set of morphisms from an object  $U$  to an object  $V$ , and  $Mor(\mathcal{C}) = \bigcup_{U, V \in Ob(\mathcal{C})} Hom(U, V)_{\mathcal{C}}$  to denote the total set of morphisms.

Recall from [17] that a small category  $\mathcal{C}$  is called a *complex algebroid* if each morphism set is a complex vector space, and composition between morphism

spaces

$$\text{Hom}(V, W)_{\mathcal{C}} \times \text{Hom}(U, V)_{\mathcal{C}} \rightarrow \text{Hom}(U, W)_{\mathcal{C}}$$

is bilinear.

If  $X$  is a real or complex vector space, recall (see [21]) that a *seminorm* on  $V$  is a map  $p: X \rightarrow \mathbb{R}$  such that:

- $p(x) \geq 0$  for all  $x \in X$ .
- $p(\lambda x) = |\lambda|p(x)$  for all  $x \in X$  and  $\lambda \in \mathbb{C}$ .
- $p(x + y) \leq p(x) + p(y)$  for all  $x, y \in X$ .

If  $V$  has a family  $\{p_\alpha \mid \alpha \in A\}$  of seminorms, we equip  $V$  with the weakest topology under which each map  $x \mapsto p_\alpha(x - x_0)$  is continuous, where  $x_0 \in V$  and  $\alpha \in A$ . With this topology,  $V$  is a *topological vector space*, that is to say the operations of addition and scalar multiplication are continuous.

A topological vector space where the topology is defined by a family of seminorms is called a *locally convex space*. The main property of locally convex spaces that we will need in these notes is the *second geometric form of the Hahn-Banach theorem*; again, see [21] for a proof.

**Theorem 2.1** *Let  $V$  be a real locally convex space. Let  $A, B \subseteq V$  be convex, with  $A$  compact,  $B$  closed and  $A \cap B = \emptyset$ . Then there is a continuous linear map  $\varphi: V \rightarrow \mathbb{R}$  and real numbers  $\alpha, \beta \in \mathbb{R}$  such that*

$$\varphi(x) \leq \alpha < \beta \leq \varphi(y)$$

for all  $x \in A$  and  $y \in B$ . □

**Definition 2.2** A complex algebroid  $\mathcal{C}$  is called a *locally convex algebroid* if each morphism set is a locally convex vector space, and composition is continuous.

Recall that a *lattice* is a partially ordered set,  $(S, \leq)$ , where any two elements  $a, b \in S$  have a *least upper bound*,  $a \vee b$ , and a *greatest lower bound*  $a \wedge b$ .

**Definition 2.3** We call an algebroid  $\mathcal{C}$  a *lattice algebroid* if:

- The set of objects has a partial ordering,  $\leq$ , under which it is a lattice.
- If  $U \leq V$  for objects  $U$  and  $V$  then there is a canonical monomorphism  $i_{U,V}: U \hookrightarrow V$  such that  $i_{V,W}i_{U,V} = i_{U,W}$  for all  $U, V, W \in \text{Ob}(\mathcal{C})$ .<sup>1</sup>
- Let  $V' \leq V$  and let  $x \in \text{Hom}(U, V)_{\mathcal{C}}$ . Let  $V' \leq V$ . Then there is an object  $x^{-1}[V'] \leq U$  and a morphism  $x|_{x^{-1}[V']}$  such that  $i_{V',V}x|_{x^{-1}[V']} = x|_{x^{-1}[V'],U}$ .

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<sup>1</sup>A morphism,  $i \in \text{Hom}(U, V)_{\mathcal{C}}$  is a *monomorphism* if it has the left-cancellation property, that is to say if  $ix = iy$  for  $x, y \in \text{Hom}(U', U)_{\mathcal{C}}$ , then  $x = y$ .

In terms of arrows, we write  $U \hookrightarrow V$  to denote the monomorphism  $i_{U,V}$  when  $U \leq V$ . For morphisms  $x \in \text{Hom}(U, V)_{\mathcal{C}}$  and  $y \in \text{Hom}(U', V')_{\mathcal{C}}$ , we write  $x \leq y$ , and call  $y$  an *extension* of  $x$  if  $U \leq U'$ , and we have an object  $W$  such that  $V, V' \leq W$  and the morphisms

$$U \xrightarrow{x} V \hookrightarrow W$$

and

$$U \hookrightarrow U' \xrightarrow{y} V' \hookrightarrow W$$

are equal.

A lattice structure lets us add morphisms in different morphism sets; this is something we will sometimes need to be able to do.

**Definition 2.4** Let  $\mathcal{C}$  be a lattice algebroid. Let  $x \in \text{Hom}(U, V)_{\mathcal{C}}$  and  $y \in \text{Hom}(U', V')_{\mathcal{C}}$ . Then we define  $x + y \in \text{Hom}(U \vee V, U \wedge V)_{\mathcal{C}}$  to be the sum of the morphisms

$$U \wedge U' \hookrightarrow U \xrightarrow{x} V \hookrightarrow V \vee V'$$

and

$$U \wedge U' \hookrightarrow U' \xrightarrow{y} V' \hookrightarrow V \vee V'.$$

We can also compose any two morphisms.

**Definition 2.5** Let  $x \in \text{Hom}(U, V)_{\mathcal{C}}$  and  $y \in \text{Hom}(U', V')_{\mathcal{C}}$ . Then we define the *product*  $yx$  to be the composite

$$x^{-1}[V \wedge U'] \xrightarrow{x|_{x^{-1}[V \wedge U']}} V \wedge U' \hookrightarrow U' \xrightarrow{T} V'$$

One warning is that the above product is not, in general, associative. Therefore, when we can, we avoid it, sticking with the associative composition of morphisms at the category level.

**Definition 2.6** Let  $\mathcal{C}$  be a locally convex algebroid. A *partial involution* on  $\mathcal{C}$  consists of:

- A set of distinguished objects,  $\text{Ob}(\mathcal{C})_0$ , called the *dense objects*. We write  $\text{Mor}(\mathcal{C})_0 = \bigcup_{U \in \text{Ob}(\mathcal{C}_0), V \in \text{Ob}(\mathcal{C})} \text{Hom}(U, V)_{\mathcal{C}}$ ,
- A function  $\text{Mor}(\mathcal{C})_0 \rightarrow \text{Mor}(\mathcal{C})_0$ , written  $x \mapsto x^*$

such that:

- For any object  $U \in \text{Ob}(\mathcal{C})$ , we have an object  $V \in \text{Ob}(\mathcal{C})_0$  such that  $U \leq V$ .
- If  $V \leq W$  and  $V$  is a dense object, then so is  $W$ , and for each object  $U$ , the map  $\text{Hom}(U, V)_{\mathcal{C}} \rightarrow \text{Hom}(U, W)_{\mathcal{C}}$  defined by the formula  $x \mapsto i_{V,W}x$  is a dense embedding.

- Let  $x, y \in \text{Hom}(U, V)_{\mathcal{C}}$ , where  $U$  is a dense object. Let  $\alpha, \beta \in \mathbb{C}$ . Then  $(\alpha x + \beta y)^* = \alpha x^* + \beta y^*$ .
- Let  $x \in \text{Hom}(U, V)_{\mathcal{C}}$  and  $y \in \text{Hom}(V, W)_{\mathcal{C}}$ , where  $U$  and  $V$  are dense objects. Then  $(yx)^* = x^*y^*$ .
- Let  $x \in \text{Mor}(\mathcal{C})_0$ . Then  $x \leq (x^*)^*$ .

A locally convex lattice algebroid with an involution is called a locally convex  $*$ -algebroid.

Note that the lattice structure on  $\mathcal{C}$  is needed to define the sum of two arbitrary morphisms, and so used to formulate the first of the above axioms.

**Definition 2.7** Let  $A$  be a unital  $C^*$ -algebra, and let  $\mathcal{C}$  be a locally convex  $*$ -algebroid. An  $A$ -bimodule structure on  $\mathcal{C}$  consists of continuous bilinear maps

$$A \times \text{Hom}(U, V)_{\mathcal{C}} \mapsto \text{Hom}(U, V)_{\mathcal{C}}$$

and

$$\text{Hom}(U, V)_{\mathcal{C}} \times A \mapsto \text{Hom}(U, V)_{\mathcal{C}}$$

written simply  $(a, x) \mapsto ax$  and  $(x, a) \mapsto xa$  respectively, under which each morphism set  $\text{Hom}(U, V)_{\mathcal{C}}$  is an  $A$ -bimodule, and

- $a(xy) = (ax)y$
- $(xa)y = x(ay)$
- $(xy)a = x(ya)$
- $(ax)^* = x^*a^*$  if  $x \in \text{Mor}(\mathcal{C})_0$ .
- Let  $1 \in A$  be the unit. Then  $1x = x1 = x$ .
- Let  $a \in A$ , and  $U \leq V$  be objects in  $\mathcal{C}$ . Then  $ai_{U,V} = i_{U,V}a$ .

The definition ensures the following is valid for the more general addition and multiplication present in a locally convex  $*$ -algebroid.

**Proposition 2.8** Let  $\mathcal{C}$  be a locally convex  $*$ -algebroid with an  $A$ -bimodule structure. Let  $x, y \in \text{Mor}(\mathcal{C})$  and  $a \in A$ . Then

$$a(x + y) = ax + ay \quad (x + y)a = xa + ya$$

and

$$a(xy) = (ax)y \quad (xa)y = x(ay) \quad (xy)a = x(ya)$$

□

Let  $\mathcal{C}$  be a locally convex  $*$ -algebroid with an  $A$ -bimodule structure. We say a morphism  $x \in \text{Hom}(U, V)_{\mathcal{C}}$  has *bounded inverse* if we have an element  $a \in A$  such that  $ax = xa = i_{U, V}$ .

We define the *spectrum* of  $T$ ,  $\text{Spectrum}(T)$ , to be the set of all  $\lambda \in \mathbb{C}$  such that the morphism  $x - \lambda i_{U, V}$  does *not* have bounded inverse.

We call an element  $x \in \text{Hom}(U, V)_{\mathcal{C}}$ , where  $U$  is a dense object, *positive* if  $x^* = x$ , and  $\text{Spectrum}(x) \subseteq [0, \infty)$ .

**Definition 2.9** Let  $A$  be a  $C^*$ -algebra. An  $LK^*$ -algebroid over  $A$  is a locally convex  $*$ -algebroid equipped with an  $A$ -bimodule structure such that for each  $x \in \text{Hom}(U, V)_{\mathcal{C}}$  we have that  $xx^*$  is positive.

$LK^*$ -algebroids have a fair amount of structure. An  $LK^*$ -functor is a functor between  $LK^*$ -algebroids that preserves all of this structure. Specifically, we have the following.

**Definition 2.10** Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $LK^*$ -algebroids over a  $C^*$ -algebra  $A$ . A function  $\gamma: \mathcal{A} \rightarrow \mathcal{B}$  is called an  $LK^*$ -functor if:

- The map  $\gamma: \text{Ob}(\mathcal{A}) \rightarrow \text{Ob}(\mathcal{B})$  is order-preserving, and takes dense objects to dense objects, with  $\gamma(U \vee V) = \gamma(U) \vee \gamma(V)$  and  $\gamma(U \wedge V) = \gamma(U) \wedge \gamma(V)$ .
- For  $U, V \in \text{Ob}(\mathcal{A})$  with  $U \leq V$ , we have  $\gamma(i_{U, V}) = i_{\gamma(U), \gamma(V)}$ .
- Let  $V' \leq V$  and let  $x \in \text{Hom}(U, V)_{\mathcal{A}}$ . Let  $V' \leq V$ . Then  $\gamma(x^{-1}[V']) = \gamma(x)^{-1}[\gamma(V')]$  and  $\gamma(x|_{x^{-1}[V']}) = \gamma(X)|_{\gamma(x)^{-1}[\gamma(V)]}$ .
- Each map  $\gamma: \text{Hom}(U, V)_{\mathcal{A}} \rightarrow \text{Hom}(\gamma(U), \gamma(V))_{\mathcal{B}}$  is continuous and linear.
- For each morphism  $T \in \text{Mor}(\mathcal{A})$  and  $a \in A$ , we have  $\gamma(aT) = a\gamma(T)$ , and  $\gamma(Ta) = \gamma(T)a$ .
- For any morphism  $T \in \text{Mor}(\mathcal{A})_0$  we have  $\gamma(T^*) = \gamma(T)^*$ .

The structure we have defined ensures that  $LK^*$ -functors preserve the more general addition and multiplication of morphisms. Specifically, we have the following.

**Proposition 2.11** Let  $\gamma: \mathcal{A} \rightarrow \mathcal{B}$  be an  $LK^*$ -functor. Then for all  $x, y \in \text{Mor}(\mathcal{A})$ , we have  $\gamma(x + y) = \gamma(x) + \gamma(y)$  and  $\gamma(xy) = \gamma(x)\gamma(y)$ .  $\square$

The following notion is slightly more general, though the above proposition still holds.

**Definition 2.12** Let  $\mathcal{A}$  an  $LK^*$ -algebroid over a  $C^*$ -algebra  $A$ , and  $\mathcal{B}$  be an  $LK^*$ -algebroid over a  $C^*$ -algebra  $B$ . A pair  $(\gamma, \theta)$ , where  $\theta: A \rightarrow B$  is an  $*$ -homomorphism, and  $\gamma: \mathcal{A} \rightarrow \mathcal{B}$  is a function, is called an  $LK^*$ -functor if:

- The map  $\gamma: Ob(\mathcal{A}) \rightarrow Ob(\mathcal{B})$  is order-preserving, and takes dense objects to dense objects, with  $\gamma(U \vee V) = \gamma(U) \vee \gamma(V)$  and  $\gamma(U \wedge V) = \gamma(U) \wedge \gamma(V)$ .
- For  $U, V \in Ob(\mathcal{A})$  with  $U \leq V$ , we have  $\gamma(i_{U,V}) = i_{\gamma(U), \gamma(V)}$ .
- Let  $V' \leq V$  and let  $x \in Hom(U, V)_{\mathcal{A}}$ . Let  $V' \leq V$ . Then  $\gamma(x^{-1}[V']) = \gamma(x)^{-1}[\gamma(V')]$  and  $\gamma(x|_{x^{-1}[V']}) = \gamma(X)|_{\gamma(x)^{-1}[\gamma(V)]}$ .
- Each map  $\gamma: Hom(U, V)_{\mathcal{A}} \rightarrow Hom(\gamma(U), \gamma(V))_{\mathcal{B}}$  is continuous and linear.
- For each morphism  $T \in Mor(\mathcal{A})$  and  $a \in A$ , we have  $\gamma(aT) = \theta(a)\gamma(T)$ , and  $\gamma(Ta) = \gamma(T)\theta(a)$ .
- For any morphism  $T \in Mor(\mathcal{A})_0$  we have  $\gamma(T^*) = \gamma(T)^*$ .

### 3 Examples

let  $H$  be a Hilbert space. Let  $U \subseteq H$  be a subset, and let  $T: U \rightarrow V \subseteq H$  be a linear map (not in general bounded); we call a not necessarily bounded linear map an *operator*.

Let  $x \in H_2$ , and define  $\varphi_y: H_1 \rightarrow \mathbb{C}$  by the formula  $\varphi_x(y) = \langle x, Ty \rangle$ . Set

$$V' = \{y \in H_2 \mid \varphi_y \text{ is continuous}\}.$$

Then one can show using the Hahn-Banach theorem (see [21]) that if  $U$  is dense, then there is a unique operator  $T^*: V' \rightarrow U' \subseteq H$  such that  $\langle T^*x, y \rangle = \langle x, Ty \rangle$  for all  $x \in W$  and  $y \in V$ . We call  $T^*$  the *adjoint* of  $T$ .

Recall that we call an operator  $T$  *closed* if the *graph*  $Gr(T) = \{(u, Tu) \mid u \in U\}$  is a closed subset of  $H \oplus H$ . This does not in general imply that  $U$  is a closed subset of  $H$ ; if this were true, by the closed graph theorem, the operator  $T$  would be bounded. See chapter 10 of [14] for details, where the following is also shown.

**Proposition 3.1** *Let  $T: U \rightarrow V$  be a closed operator. Then the above domain of the adjoint,  $V'$ , is a dense subset of  $H$ .  $\square$*

In particular, in this case, we can form the second adjoint  $(T^*)^*$ . It turns out that  $(T^*)^* = T$ ; again, see [14] for details. The definition of  $LK^*$ -categories was motivated by the following result, which is now straightforward to verify.

The locally convex topology described is typical in the literature on topological algebras of operators; see for example [15].

**Proposition 3.2** *Let  $H$  be a Hilbert spaces. Let  $\mathcal{U}(H)$  be the category where the set of objects is the collection of linear subspaces of  $H$ , and the morphisms are closed operators between them. Let us define a locally convex topology on the space  $Hom(U, V)_{\mathcal{U}(H)}$  by the family of seminorms*

$$p_M(T) = \sup\{\langle u, Tv \rangle \mid (u, v) \in M\}$$

where  $M$  is a subset of  $V \times U$  such that the above supremum is finite.

Define a partial ordering on  $Ob(\mathcal{U}(H))$  by taking subsets, and a lattice structure by writing  $U \wedge V = U \cap V$  and  $U \vee V = U + V$ . If  $U \subseteq V$ , let  $i_{U,V}$  be inclusion map  $U \hookrightarrow V$ . If  $T \in Hom(U, V)_{\mathcal{U}(H)}$ , and  $V' \subseteq V$ , let  $T^{-1}[V'] = \{u \in U \mid T(u) \in V'\}$ , and let  $T|_{T^{-1}[V']}: T^{-1}[V'] \rightarrow V'$  be defined by restricting  $T$  to this set.

Call  $U \in Ob(\mathcal{U}(H))$  a dense object if  $U$  is a dense subset of  $H$ . If  $T \in Hom(U, V)_{\mathcal{C}}$  for some subspace  $V$ , define  $T^*$  to be the above adjoint.

Finally, let  $\mathcal{B}(H)$  denote the  $C^*$ -algebra of bounded linear operators on  $H$ . Then  $\mathcal{U}(H)$  is an  $LK^*$ -algebroid over  $\mathcal{B}(H)$ .  $\square$

**Definition 3.3** Let  $A$  be a  $C^*$ -algebra, and let  $B$  be a sub-algebra of  $A$ . We call a subalgebroid,  $\mathcal{D}$ , of an  $LK^*$ -category,  $\mathcal{C}$ , over  $A$ , a *sub- $LK^*$ -algebroid over  $B$*  if:

- The greatest lower and least upper bounds of any two objects in  $\mathcal{D}$  are also in  $\mathcal{D}$ .
- Let  $U \leq V$ , where  $U, V \in Ob(\mathcal{D})$ . Then  $i_{U,V} \in Hom(U, V)_{\mathcal{D}}$ .
- Let  $x \in Hom(U, V)_{\mathcal{D}}$ , and let  $V' \in Ob(\mathcal{D})$  be such that  $V' \leq V$ . Then  $x^{-1}[V'] \in Ob(\mathcal{D})$  and  $x|_{x^{-1}[V']} \in Hom(x^{-1}[V'], V')_{\mathcal{D}}$ .
- Each morphism set  $Hom(U, V)_{\mathcal{D}}$  is a  $B$ -bimodule, with operations inherited from the  $A$ -bimodule  $Hom(U, V)_{\mathcal{C}}$ .
- Let  $U \in Ob(\mathcal{D})$ . Then there is an object  $V \in Ob(\mathcal{D})$  such that  $U \leq V$  and  $V$  is a dense object in  $\mathcal{C}$ .
- Let  $x \in Hom(U, V)_{\mathcal{D}}$ , where  $U$  is a dense object in  $\mathcal{C}$ . Then  $x^* \in Mor(\mathcal{D})$ .

Certainly, a sub- $LK^*$ -algebroid over  $B$  of an  $LK^*$ -algebroid is itself an  $LK^*$ -algebroid with its inherited structure.

**Example 3.4** Let  $H$  be a Hilbert space, let  $U \subseteq H$  be a dense subspace, let  $V$  be another subspace, and let  $D: U \rightarrow V$  be a closed operator. Then we write  $\mathcal{U}^*(D)$  to denote the smallest sub- $LK^*$ -algebroid over  $\mathbb{C}$  of  $\mathcal{U}(H)$  that contains the objects  $U$  and  $V$  and the operator  $D \in Hom(U, V)_{\mathcal{U}^*(D)}$ .

Recall (see [13]) that an algebroid  $\mathcal{C}$  is called a  $C^*$ -category if each morphism set is a Banach algebra, and:

- Composition of morphisms satisfies the inequality

$$\|xy\| \leq \|x\| \cdot \|y\| \quad x \in Hom(V, W)_{\mathcal{C}}, y \in Hom(U, V)_{\mathcal{C}}$$

- There are conjugate linear maps  $Hom(U, V)_{\mathcal{C}} \rightarrow Hom(V, U)_{\mathcal{C}}$ , written  $x \mapsto x^*$  such that  $(xy)^* = y^*x^*$  if  $x$  and  $y$  are composable morphisms, and  $(x^*)^* = x$  for any morphism  $x$ .



- The  $C^*$ -identity  $\|xx^*\| = \|x\|^2$  holds for any morphism  $x$ .
- If  $x \in \text{Hom}(U, V)_{\mathcal{C}}$ , then the composite  $xx^*$  is a positive element of the  $C^*$ -algebra  $\text{Hom}(V, V)_{\mathcal{C}}$

We call a  $C^*$ -category *additive* if there is a 0 object, 0, and for any two objects  $U$  and  $V$  there is a biproduct (in the sense of category theory; see for example [16, 23])  $U \oplus V$ . As shown in [18], any  $C^*$ -category  $\mathcal{C}$  has an *additive completion*  $\mathcal{C}_{\oplus}$ . Objects of the additive completion  $\mathcal{C}_{\oplus}$  are formal strings

$$U_1 \oplus U_2 \oplus \cdots \oplus U_m \quad U_i \in \text{Ob}(\mathcal{C}).$$

Let us write

$$(U_1 \oplus U_2 \oplus \cdots \oplus U_m) \vee (V_1 \oplus V_2 \oplus \cdots \oplus V_n) = U_1 \oplus U_2 \oplus \cdots \oplus U_m \oplus V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

and let  $(U_1 \oplus U_2 \oplus \cdots \oplus U_m) \wedge (V_1 \oplus V_2 \oplus \cdots \oplus V_n)$  be the largest string  $W_1 \oplus \cdots \oplus W_r$  such that:

- $W_i = U_{a_i} = V_{b_i}$  for some  $a_i$  and  $b_i$ .
- If  $i \leq j$ , then  $a_i \leq a_j$  and  $b_i \leq b_j$ .

If no such string exists, we set  $(U_1 \oplus U_2 \oplus \cdots \oplus U_m) \wedge (V_1 \oplus V_2 \oplus \cdots \oplus V_n) = 0$ . The following is then straightforward to check.

**Proposition 3.5** *Let  $\mathcal{C}$  be a  $C^*$ -category. For  $U, V \in \text{Ob}(\mathcal{C}_{\oplus})$ , write  $U \leq V$  if  $U \vee V = U$ . Then, with the above lattice structure on the objects,  $\mathcal{C}_{\oplus}$  is an  $LK^*$ -category in which every object is dense.  $\square$*

We conclude our examples by looking at how to make new  $LK^*$ -categories out of old ones. The first construction is fairly obvious.

**Proposition 3.6** *Let  $A$  be a  $C^*$ -algebra. Let  $\mathcal{C}$  and  $\mathcal{D}$  be  $LK^*$ -categories over  $A$ . Then we have an  $LK^*$ -category  $\mathcal{C} \oplus \mathcal{D}$  over  $A$  where:*

- $\text{Ob}(\mathcal{C} \oplus \mathcal{D})$  is the set of formal pairs  $U \oplus U'$  where  $U \in \text{Ob}(\mathcal{C})$  and  $U' \in \text{Ob}(\mathcal{D})$ .

•

$$\text{Hom}(U \oplus U', V \oplus V')_{\mathcal{C} \oplus \mathcal{D}} = \text{Hom}(U, V)_{\mathcal{C}} \oplus \text{Hom}(U', V')_{\mathcal{D}}$$

**Proof:** Say  $U \oplus U' \leq V \oplus V'$  if  $U \leq U'$  and  $V \leq V'$ . Then we have a lattice structure defined by writing

$$U \oplus U' \vee V \oplus V' = (U \vee V) \oplus (U' \vee V') \quad U \oplus U' \wedge V \oplus V' = (U \wedge V) \oplus (U' \wedge V').$$

We can write  $i_{U \oplus U', V \oplus V'} = i_{U, V} \oplus i_{U', V'}$ . If  $x \in \text{Hom}(U, V)$  and  $y \in \text{Hom}(U', V')$ , with  $W \leq V$  and  $W' \leq V'$ , we can define

$$(x \oplus y)^{-1}[W \oplus W'] = x^{-1}[W] \oplus y^{-1}[W'] \quad (x \oplus y)(x \oplus y)^{-1}[W \oplus W'] = x|_{x^{-1}[W]} \oplus y|_{y^{-1}[W']}$$

Call  $U \oplus U'$  dense if  $U$  is dense in  $\mathcal{C}$  and  $U'$  is dense in  $\mathcal{D}$ . Given  $x \oplus y \in \text{Hom}(U \oplus U', V \oplus V')_{\mathcal{C} \oplus \mathcal{D}}$ , define  $(x \oplus y)^* = x^* \oplus y^*$ .

Then the required axioms are easy to check.  $\square$

The following is similar.

**Proposition 3.7** *Let  $A$  and  $B$  be  $C^*$ -algebras. Let  $\mathcal{C}$  be an  $LK^*$ -category over  $A$ , and  $\mathcal{D}$  be an  $LK^*$ -category over  $B$ . Then we have an  $LK^*$ -category  $\mathcal{C} \oplus \mathcal{D}$  over  $A \oplus B$  where:*

- *$\text{Ob}(\mathcal{C} \oplus \mathcal{D})$  is the set of formal pairs  $U \oplus U'$  where  $U \in \text{Ob}(\mathcal{C})$  and  $U' \in \text{Ob}(\mathcal{D})$ .*

•

$$\text{Hom}(U \oplus U', V \oplus V')_{\mathcal{C} \oplus \mathcal{D}} = \text{Hom}(U, V)_{\mathcal{C}} \oplus \text{Hom}(U', V')_{\mathcal{D}}$$

$\square$

We conclude by looking at tensor products. Let  $V$  and  $W$  be locally convex vector spaces over  $\mathbb{C}$  with topologies defined by the families of seminorms  $\{p_a \mid a \in A\}$  and  $\{q_b \mid b \in B\}$  respectively. Recall (see for example [21]) that we can define a family of seminorms  $\{p_a \otimes q_b \mid a \in A, b \in B\}$  on the tensor product  $V \otimes W$  by the formula

$$p_a \otimes q_b(x) = \inf \left\{ \max_{j=1}^n p_a(u_j) \cdot q_b(v_j) \mid x = \sum_{j=1}^n u_j \otimes v_j \right\}.$$

We call the locally convex topology on  $V \otimes W$  defined by this set of seminorms the *projective topology*.

**Proposition 3.8** *Let  $\mathcal{C}$  be an  $LK^*$ -category over a  $C^*$ -algebra  $A$ . Let  $B$  be another  $C^*$ -algebra. Then we have an  $LK^*$ -category  $\mathcal{C} \otimes B$  over  $A \otimes B$  with objects, dense objects and lattice structure the same as in  $\mathcal{C}$ , and morphism sets*

$$\text{Hom}(U, V)_{\mathcal{C} \otimes B} = \text{Hom}(U, V)_{\mathcal{C}} \otimes B$$

*equipped with the projective topology.*

**Proof:** We can define an  $A \otimes B$ -bimodule structure on a morphism set  $\text{Hom}(U, V)_{\mathcal{C} \otimes B}$  by writing

$$(a \otimes b)(x \otimes b') = ax \otimes bb' \quad (x \otimes b')(a \otimes b) = xa \otimes b'b$$

If  $U \in \text{Ob}(\mathcal{C})_0$ , and  $x \in \text{Hom}(U, V)_{\mathcal{C}}$ ,  $b \in B$ , we define  $(x \otimes b)^* = x^* \otimes b^*$ . The required axioms are easy to check.  $\square$

**Example 3.9** Consider the unbounded operator  $\frac{d}{dx}$  on  $L^2(\mathbb{R})$ . Then we define the  $LK^*$ -algebroid of differential operators on  $\mathbb{R}$ ,  $\Psi(\mathbb{R})$ , to be the tensor product

$$\mathcal{U}^* \left( \frac{d}{dx} \right) \otimes C_0(\mathbb{R}).$$

The above is a foundation for further examples of  $LK^*$ -algebroids of differential operators.

## 4 States and Representations

**Definition 4.1** Let  $\mathcal{C}$  be an  $LK^*$ -algebroid over a  $C^*$ -algebra  $A$ . Then a *representation* if  $\mathcal{C}$  is an  $LK^*$ -functor  $(\rho, \theta): (\mathcal{C}, A) \rightarrow (\mathcal{U}(H), \mathcal{B}(H))$  for some Hilbert space  $H$ . We call  $\rho$  *faithful* if it is injective on each morphism set.

The major result of this section is that any  $LK^*$ -algebroid has a faithful representation. This is a generalisation of the corresponding result for  $C^*$ -algebras (see [12]) through the well-known GNS construction. Indeed, the proof is conceptually very similar to the  $C^*$ -algebra result, and its generalisation to  $C^*$ -categories in [13, 19]

First of all, let  $V \in \text{Ob}(\mathcal{C})$ . Let  $M_V$  be the direct limit of the locally convex vector spaces  $\text{Hom}(U, V)_{\mathcal{C}}$ , where  $U \in \text{Ob}(\mathcal{C})_0$ , and if  $U' \leq U$ , we can identify  $x \in \text{Hom}(U, V)_{\mathcal{C}}$  with  $x i_{U', U} \in \text{Hom}(U', V)_{\mathcal{C}}$ . This limit makes sense, and is a vector space because of the lattice algebroid structure.

Similarly, if  $V \leq V'$ , we identify  $x \in \text{Hom}(U, V)_{\mathcal{C}}$  with  $i_{V, V'} x \in \text{Hom}(U, V')_{\mathcal{C}}$ . Similarly, if  $U' \leq U$ , we can identify  $x \in \text{Hom}(U, V)_{\mathcal{C}}$  with  $x i_{U', U} \in \text{Hom}(U', V)_{\mathcal{C}}$ . The following therefore makes sense.

**Definition 4.2** Let  $\mathcal{C}$  be an  $LK^*$ -algebroid over a  $C^*$ -algebra  $A$ . Let  $V \in \text{Ob}(\mathcal{C})_0$ . Then a *state* on  $V$  is a continuous linear map  $\sigma: M_V \rightarrow \mathbb{C}$  such that  $\sigma(1_V) = 1$  for the identity  $1_V \in \text{Hom}(V, V)_{\mathcal{C}}$ , and  $\sigma(p) \geq 0$  if  $p \in M_V$  is positive.

In particular, note that  $\sigma(xx^*) \geq 0$  for all  $x \in M_V$ .

**Proposition 4.3** Let  $x, y \in M_U$ . Then  $\sigma(xy^*) = \overline{\sigma(yx^*)}$ .

**Proof:** Let  $\lambda \in \mathbb{C}$ . Then we know that

$$0 \leq \sigma((x + \lambda y)(x + \lambda y)^*) = \sigma(xx^*) + |\lambda|^2 \sigma(yy^*) + \lambda \sigma(yx^*) + \bar{\lambda} \sigma(xy^*)$$

Now the sum  $\sigma(xx^*) + |\lambda|^2 \sigma(yy^*)$  is a real number so the sum  $\lambda \sigma(yx^*) + \bar{\lambda} \sigma(xy^*)$  is also real. Taking  $\lambda = 1$ , and  $\lambda = i$ , we see, respectively, that

$$\text{Im } \sigma(yx^*) = -\text{Im } \sigma(xy^*) \quad \text{Re } \sigma(yx^*) = \text{Re } \sigma(xy^*).$$

The result now follows. □

**Proposition 4.4** *Let  $x, y \in M_V$ . Then*

$$|\sigma(xy^*)|^2 \leq \sigma(xx^*)\sigma(yy^*).$$

**Proof:** The result is obvious if  $\sigma(xy^*) = 0$ . So let  $\sigma(xy^*) \neq 0$ . Let  $\lambda \in \mathbb{R}$ , and define

$$\alpha = \frac{\lambda\sigma(xy^*)}{|\sigma(xy^*)|}.$$

By the above  $\sigma((x + \alpha y)(x + \alpha y)^*) \geq 0$ , so

$$\lambda^2\sigma(yy^*) + 2\lambda|\sigma(xx^*)| + \sigma(xx^*) \geq 0$$

for all  $\lambda \in \mathbb{R}$ . Consideration of the discriminant of this quadratic yields the desired result.  $\square$

For a state  $\sigma$  on  $V$ , set

$$N_V = \{x \in M_V \mid \sigma(xx^*) = 0\}$$

and let  $\pi: M_V \rightarrow M_V/N_V$  be the quotient map. Then by the above two propositions, we have an inner product on the space  $M_V/N_V$  defined by the formula

$$\langle \pi(x), \pi(y) \rangle = \sigma(yx^*).$$

We can complete the quotient space  $M_V/N_V$  to obtain a Hilbert space  $H_V$ .

**Lemma 4.5** *Let  $\mathcal{C}$  be an  $LK^*$ -algebroid over a  $C^*$ -algebra  $A$ . Let  $V \in \text{Ob}(\mathcal{C})_0$ . Let  $\sigma$  be a state on  $V$ . Then for all  $x \in M_V$  and  $a \in A$  we have*

$$\|\pi(xa)\| \leq \|\pi(x)\| \cdot \|a\|.$$

**Proof:** Set

$$b = \frac{aa^*}{\|a\|^2}.$$

Then  $\|b\| = 1$ , so  $1 - b$  is positive. Hence, by functional calculus on the  $C^*$ -algebra  $A$ , we have  $c \in A$  such that  $c^2 = 1 - b$ . Observe

$$(cx)(cx)^* = x(1 - b)x^*$$

so

$$\sigma(x(1 - b)x^*) \geq 0$$

from which it follows that

$$\sigma(xx^*) \geq \sigma(xbx^*) = \sigma\left(\frac{xaa^*x^*}{\|a\|^2}\right)$$

and the result follows.  $\square$

Similarly

$$\|\pi(ax)\| \leq \|a\| \cdot \|\pi(x)\|.$$

It follows that we have a representation  $\theta: A \rightarrow \mathcal{B}(H_V)$  defined by writing

$$\theta(a)(\pi(x)) = \pi(ax).$$

**Theorem 4.6** *Let  $\mathcal{C}$  be an  $LK^*$ -category over a  $C^*$ -algebra  $A$ , and let  $U \in \text{Ob}(\mathcal{C})_0$ . Let  $\sigma$  be a state on  $U$ . Then there is a representation  $\rho: \mathcal{C} \rightarrow \mathcal{U}(H)$  for some Hilbert space  $H$ , and an element  $u \in H$  such that  $\|u\| = 1$ , and*

$$\sigma(x) = \langle u, \rho(x)u \rangle$$

for all  $x \in M_U$ .

**Proof:** Let  $U \leq W$ . Define a state on  $V$  by  $\sigma_W(x) = \sigma(x|_{x^1[W]})$  if  $x \in M_W$ . Then we can form a Hilbert space  $H_W$  by the above process. Let  $H = \lim_{U \leq W} H_W$ . We have a representation  $\theta: A \rightarrow \mathcal{B}(H)$  defined as above.

By the second axiom for the partial involution, for any dense object  $V$ ,  $\pi[M_V]$  is a dense subset of  $H$ . Let  $\rho(V) = \pi[M_V]$ . Let  $x \in \text{Hom}(V, W)_{\mathcal{C}}$ . Then we have an operator  $M_x: \pi(M_V) \rightarrow \pi(M_W)$  defined by the formula

$$M_x \pi(y) = \pi(xy).$$

Let  $(u_n, M_x u_n) \rightarrow (u, v)$  as  $n \rightarrow \infty$ , with respect to the norm on  $H_V \oplus H_W$  defined by the inner product. Set  $u_n = \pi(y_n)$ ,  $u = \pi(y)$ , and  $v = \pi(z)$ . Then we have

$$(\pi(y_n), \pi(xy_n)) \rightarrow (\pi(y), \pi(z))$$

as  $n \rightarrow \infty$ , that is to say

$$\sigma((y_n - y)(y_n - y)^*) \rightarrow 0 \quad \sigma((xy_n - z)(xy_n - z)^*) \rightarrow 0$$

as  $n \rightarrow \infty$ .

From the first of these and proposition 4.4, we see that  $\sigma((xy_n - xy)(xy_n - xy)^*) \rightarrow 0$  as  $n \rightarrow \infty$ . Combining this with the second of the above limits, we see that  $xy - z \in N_V$ , and hence that  $(u, v)$  belongs to the graph of  $M_x$ . In other words, we have shown that  $M_x$  is a closed operator.

Let  $1_U \in \text{Hom}(U, U)_{\mathcal{C}}$  be the identity. Set  $u = \pi(1_U)$ . Then

$$\|u\|^2 = \sigma(1) = 1$$

and

$$\langle u, \rho(x)u \rangle = \langle \pi(1_U), \pi(x1_U) \rangle = \sigma(x).$$

It is now routine to check that  $(\rho, \theta)$  is a representation of  $\mathcal{C}$ . □

**Lemma 4.7** *Let  $\mathcal{C}$  be an  $LK^*$ -category, and let  $U$  be a dense object. Let  $x \in M_U$ ,  $x \neq 0$ . Then we have a state,  $\sigma$ , on  $U$  such that  $\sigma(xx^*) > 0$ .*

**Proof:** Let  $M_U^+$  be the set of positive elements of the locally convex vector space  $M_U$ . Let  $M_U^{\mathbb{R}}$  be the smallest real vector space containing  $M_U^+$ . Then  $M_U^+$  is a closed convex subspace of  $M_U^{\mathbb{R}}$ . Hence, by the second geometric form of the Hahn-Banach theorem, we have a continuous linear map  $\varphi: M_U^{\mathbb{R}} \rightarrow \mathbb{R}$  and real numbers  $\alpha, \beta \in \mathbb{R}$  such that

$$\varphi(-xx^*) < \beta < \varphi(y)$$

for all  $y \in M_U^+$ .

Taking  $y = 0$ , we see that  $\beta < 0$ , so  $\varphi(xx^*) > 0$ . Suppose  $\varphi(y) < 0$  for some  $y \in M_U^+$ . Let  $\lambda = \frac{\beta}{\varphi(y)} > 0$ . Then  $\lambda y$  is positive, and  $\varphi(\lambda y) = \beta$ , which contradicts the above inequality. Therefore  $\varphi(y) \geq 0$  for all  $y \in M_U^+$ .

Let  $\psi(z) = \frac{1}{\varphi(1)}\varphi(z)$  for  $z \in M_U^{\mathbb{R}}$ . Then  $\psi(y) \geq 0$  if  $y$  is positive,  $\psi(1) = 1$ , and  $\psi(xx^*) > 0$ .

Extend  $\psi$  to a complex linear functional  $\tilde{\psi}: M_U^{\mathbb{R}} + iM_U^{\mathbb{R}} \rightarrow \mathbb{C}$  by the formula

$$\tilde{\psi}(u + iv) = \psi(u) + i\psi(v).$$

Then by the Hahn-Banach theorem there is a continuous linear extension  $\sigma: M_U \rightarrow \mathbb{C}$  of  $\tilde{\psi}$ . By construction,  $\sigma$  is a state with  $\sigma(xx^*) > 0$ .  $\square$

**Theorem 4.8** *Let  $\mathcal{C}$  be an  $LK^*$ -category over a  $C^*$ -algebra  $A$ . Then we have a faithful representation  $(\rho, \theta)$ .*

**Proof:** Pick  $V \in \text{Ob}(\mathcal{C})_0$ . Let  $\sigma$  be a state on  $V$ . Then by the above, we have a representation  $(\rho_\sigma, \theta_\sigma)$  on a Hilbert space  $H$ , and a vector  $u \in H$  such that

$$\sigma(x) = \langle u, \rho(x)u \rangle$$

for all  $x \in M_V$ .

Let  $\Sigma$  be the set of states on  $U$ . Let

$$\rho_V = \bigoplus_{\sigma \in \Sigma} \rho_\sigma \quad \theta_V = \bigoplus_{\sigma \in \Sigma} \theta_\sigma.$$

Let  $x \in \text{Hom}(U, V)_{\mathcal{C}}$ . Then by the above lemma, we have a state  $\sigma$  with  $\sigma(xx^*) > 0$ . It follows that  $\rho(x) \neq 0$ .

Hence, if we define

$$\rho = \bigoplus_{V \in \text{Ob}(\mathcal{C})_0} \rho_V \quad \theta = \bigoplus_{V \in \text{Ob}(\mathcal{C})_0} \theta_V$$

then  $(\rho, \theta)$  is a faithful representation, and we are done.  $\square$

## References

- [1] An algebraic approach to quantum field theory. *Journal of Mathematical Physics*, 5:848–861, 1964.
- [2] J.-P. Antoine. Partial  $*$ -algebras, a tool for the mathematical description of physical systems? In S. Twareque Ali and Kalyan B. Sinha, editors, *Contributions in mathematical physics: A tribute to Gerard G. Emch*, pages 37–68. Hindustan Book Agency, New Delhi, 2007.
- [3] J.-P. Antoine, A. Inoue, and C. Trapani. Biweights on partial  $*$ -algebras, 2000.

- [4] J.-P. Antoine, A. Inoue, and C. Trapani. *Partial  $*$ -algebras and their operator realizations*. Kluwer, Dordrecht, 2002.
- [5] J.-P. Antoine and W. Karwowski. Partial *ast*-algebras of closed linear operators in hilbert space. *Publications of the Research Institute for Mathematical Sciences, Kyoto University*, 21:205–236, 1985.
- [6] J.-P. Antoine, F. Bagarallo, and C. Trapani. Topological partial  $*$ -algebras: Basic properties and examples. *Reviews in Mathematical Physics*, 11:267–302, 1999.
- [7] F. Bagarello. Algebras of unbounded operators and physical applications: a survey. *Reviews in Mathematical Physics*, 19:231–271, 2007.
- [8] A. Connes. *Noncommutative Geometry*. Academic Press, 1994.
- [9] K.R. Davidson.  *$C^*$ -algebras by example*, volume 6 of *Fields Institute Monographs*. American Mathematical Society, 1996.
- [10] J. Dixmier.  *$C^*$ -algebras (translated from the French by Francis Jellet)*, volume 15 of *North Holland Mathematical Library*. Elsevier, 1983.
- [11] G.G. Emch. *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*. Wiley-Interscience, New York, 1972.
- [12] I. Gelfand and M. Neumark. On the imbedding of normed rings into the ring of operators in Hilbert space. In S. Doran, editor,  *$C^*$ -algebras: 1943–1993*, volume 167 of *Contemporary Mathematics*, pages 2–19. American Mathematical Society, 1994.
- [13] P. Ghez, R. Lima, and J. Roberts.  $W^*$ -categories. *Pacific Journal of Mathematics*, 120:79–109, 1985.
- [14] E. Kreyszig. *introductory functional analysis with applications*. John Wiley and sons, 1987.
- [15] G. Lassner. Topological algebras of operators. *Reports on Mathematical Physics*, 3:279–293, 1972.
- [16] S. MacLane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer, Berlin, 1998.
- [17] B. Mitchell. *Separable algebroids*, volume 333 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 1985.
- [18] P.D. Mitchener. Symmetric  $K$ -theory spectra of  $C^*$ -categories.  *$K$ -theory*, 24:157–201, 2001.
- [19] P.D. Mitchener.  $C^*$ -categories. *Proceedings of the London Mathematical Society*, 84:374–404, 2002.

- [20] G.J. Murphy. *C\*-algebras and operator theory*. Academic Press, 1990.
- [21] W. Rudin. *Functional Analysis*. McGraw-Hill, 1991.
- [22] States and representations of  $CQ^*$ -algebras. *Annales de l'Institut Henri Poincaré*, 61:103–133, 1994.
- [23] C.A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1994.