# Solutions to Functional Analysis 2016–2017 exam

## Question 1

### Part (i)

A normed vector space is a Banach space if it is complete, that is to say all Cauchy sequences converge to limits within the space.

### Part (ii)

Let $(f_n)$ be a Cauchy sequence in $C(X)$. Let $x \in X$. Then by definition of the norm on $C(X)$, for each $m, n$ we have that

$$|f_m(x) - f_n(x)| \leq \|f_m - f_n\|.$$

Hence the sequence $(f_n(x))$ is a Cauchy sequence of complex numbers. Since the complex numbers are complete, it follows that this sequence converges to some limit. Call this limit $f(x)$.

Let $\varepsilon > 0$. Then since $(f_n)$ is Cauchy, we have $N$ such that if $m, n \geq N$ then for each $x \in X$, we have

$$|f_m(x) - f_n(x)| \leq \|f_m - f_n\| < \frac{\varepsilon}{2}.$$

Let $n \to \infty$. Then we see that for all $x \in X$, we have $|f_m(x) - f(x)| \leq \frac{\varepsilon}{2}$ whenever $m \geq N$. Hence, for $m \geq N$, we have

$$\|f_m - f\| \leq \frac{\varepsilon}{2} < \varepsilon.$$

So the sequence $(f_n)$ converges to the function $f$. It remains to prove that $f \in C(X)$, that is to say $f$ is continuous.

Let $\varepsilon > 0$. Let $x_0 \in X$. Then:

- We can pick $N$ such that $\|f_n - f\| < \frac{\varepsilon}{3}$ whenever $n \geq N$.
- We can pick $\delta > 0$ such that $|f_N(x) - f(x)| < \delta$ whenever $d(x, x_0) < \delta$.

Hence, for $d(x, x_0) < \delta$ we have

$$|f(x_0) - f(x)| \leq |f(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(x)| + |f_N(x) - f(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

So the function $f$ is continuous and we are done.

### Part (iii)

A subset $A \subseteq V$ is closed if for any sequence $(x_n)$ in $A$ converging to a limit $x \in V$, we have that $x \in A$.

## Part (iv)

Let $A$ be a subalgebra of $C(X)$ which separates points, contains units, and satisfies the property that if $f \in A$ then $f^* inA$, where $f^*(x) = \overline{f(x)}$. Then $A$ is dense in $C(X)$.

## Part (v)

1. Let $V$ be the set of polynomials of degree 3. Then $V$ is finite-dimensional, and therefore complete. In particular, $V$ must be closed.
2. By the Stone-Weierstrass theorem, the set of all polynomials is dense $C([0,1])$, that is to say any function $f \in C([0,1])$ is the limit of a sequence of polynomials. Hence the set of polynomials is not closed.
3. The evaluation map $E\colon C([0,1]) \to \mathbb{C}$ defined by the formula $E(f) = f(0)$ is continuous. Hence the kernel of $E$ is closed. The kernel of $E$ is the required setl $\{f \in C([0,1]) \mid f(0) = 0\}$.

# Question 2

## Part (i)

The linear map $f\colon V \to \mathbb{C}$ is a bounded linear map if we have $M \geq 0$ such that

$$|f(x)| \leq M\|x\|$$

for all $x \in V$. We define the norm

$$\|f\| = \sup x \in V\backslash\{0\}\frac{|f(x)|}{\|x\|}$$

## Part (ii)

1. Observe
$$|I(f)| = \left|\int_0^1 f(t)\ dt\right| \leq \int_0^1 |f(t)|\ dt \leq \int_0^1 \|f\|\ dt = \|f\|.$$
So $I$ is bounded and $\|I\| \leq 1$. Consider the constant function defined by the formula $f(t) = 1$ for all $t \in [0,1]$. Then $\|f\| = 1$ and $|I(f)| = 1$. Hence $\|I\| = 1$.
2. Observe
$$|E(f)| = 2|f(0)| \leq 2\|f\|.$$
So $E$ is bounded, and $\|E\| \leq 2$. Using the same function as in (a), we see that $\|E\| = 2$.
3. Let $f_n(x) = x^n$. Then $\|f_n\| = 1$, and $f_n'(x) = nx^{n-1}$ so $|D(f_n)| = n$. So if $D$ were bounded, we would have $\|D\| \geq n$ for all $n$. This is clearly impossible, so $D$ is not bounded.

## Part (iii)

Let $V$ be a normed vector space, and let $W$ be a vector subspace of $V$. Let $f\colon W \to \mathbb{K}$ be a bounded linear map. Then we have a bounded linear map $F\colon V \to \mathbb{K}$ such that $F|_W = f$ and $\|F\| = \|f\|$.

## Part (iv)

Let $(f_n)$ be a Cauchy sequence in $V^*$. Let $x \in V$. Then by definition of the norm on $V^*$, for each $m, n$ we have that

$$|f_m(x) - f_n(x)| \leq \|f_m - f_n\|\|x\|.$$

Hence the sequence $(f_n(x))$ is a Cauchy sequence of complex numbers. Since the complex numbers are complete, it follows that this sequence converges to some limit. Call this limit $f(x)$.

By the algebra of limits, the function $f$ is linear. Since the sequence $(f_n)$ is Cauchy, it is bounded, so we have $M \geq 0$ such that $\|f_n\| \leq M$ for all $n$. Hence for each $x \in X$, we have that $|f_n(x)| \leq M\|x\|$ for all $n$. If we let $n \to \infty$, we see $|f(x)| \leq M\|x\|$ for all $x \in V$, so $f$ is a bounded linear map.

Let $\varepsilon > 0$. Then since $(f_n)$ is Cauchy, we have $N$ such that if $m, n \geq N$ then for each $x \in V$, we have

$$|f_m(x) - f_n(x)| \leq \|f_m - f_n\|\|x\| < \frac{\varepsilon}{2}\|x\|.$$

Let $n \to \infty$. Then we see that for all $x \in V$, $x \neq 0$ we have

$$\frac{|f_m(x) - f(x)|}{\|x\|} \leq \frac{\varepsilon}{2} < \varepsilon$$

whenever $m \geq N$.

So the sequence $(f_n)$ converges to the function $f$.

## Part (v)

Let $v \in V$. Observe

$$\|\tau(v)(f)\| = \|f(v)\| \leq \|f\| \cdot \|v\|$$

for all $f \in V^*$.

It follows that $\|\tau(v) \leq \|v\|$.

If $v = 0$, then $\tau(v) = 0$, so $\|\tau(v)\| = \|v\|$. So suppose $\tau(v) \neq 0$.

Let $W$ be the one-dimensional vector space spanned by the vector $v$. Define $f\colon W \to \mathbb{K}$ by

$$f(\alpha v) = \alpha\|v\| \qquad \alpha \in \mathbb{K}.$$

Then by the Hahn-Banach theorem, we have $F \in V^*$ such that $\|F\| = \|f\|$, and $F(v) = f(v) = 1$.

We see that

$$\|F\| = \|f\| = \sup_{\alpha \neq 0} \frac{|f(\alpha v)|}{|\alpha| \cdot \|v\|} = 1.$$

Hence

$$\|\tau(v)\| \geq |\tau(v)(F)| = |F(v)| = \|v\|.$$

Combining the two inequalities, we are done.

# Question 3

## Part (i)

We define the adjoint, $T^* \colon H \to H$, by the formula

$$\langle T^* u, v \rangle = \langle u, Tv \rangle$$

for all $u, v \in H$.

Suppose we have another map $S \colon H \to H$ such that $\langle Su, v \rangle = \langle u, Tv \rangle$ for all $u, v \in H$. Then for each $u \in H$ we have

$$\langle Su, v \rangle = \langle T^* u, v \rangle$$

for all $v \in H$, that is to say

$$\langle Su - T^* u, v \rangle = 0$$

for all $v \in H$. It follows that $Su - T^* u = 0$, that is $Su = T^* u$. So the adjoint $T^*$ is uniquely defined by the formula.

## Part (ii)

Let $f \in L^2(\mathbb{R})$. Let $M = \sup\{|k(s,t)| \mid s, t \in \mathbb{R}\}$. Observe

$$\|T_k(f)\|^2 = \int_{-\infty}^{\infty} |k(s,t)|^2 |f(t)|^2 \, dt \leq M^2 \|f\|^2.$$

So $\|T_k(f)\| \leq M\|f\|$. In particular, $T_k(f) \in L^2(\mathbb{R})$, and $T_k$ is bounded, with $\|T_k\| \leq M$.

Define $S_k \colon L^(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$S_k(g)(s) = \int_{-\infty}^{\infty} \overline{k(s,t)} g(t) \, dt$$

Observe

$$\langle S_k(f), g \rangle = \int_{\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\overline{k(t,s)} f(t)} g(s) \, ds \, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{f(t)} k(t,s) g(s) \, dt \, ds = \langle f, T_k(g) \rangle.$$

## Part (iii)

Let $V$ and $W$ be Banach spaces. Let $T\colon V \to W$ be a linear map such that the graph of $T$,

$$Graph(T) = \{(v, Tv) \mid v \in V\} \subseteq V \oplus W$$

is closed. Then $T$ is a bounded linear map.

## Part (iv)

We will use the closed graph theorem. Let $(u, v) \in \overline{Graph(T)}$. Then we have a sequence $(u_n, v_n)$ in $Graph(T)$ converging to $(u, v)$. Since $(u_n, v_n) \in Graph(T)$, we know that $v_n = Tu_n$.

Let $w \in H$. Then

$$\langle S(w), u_n\rangle = \langle w, Tu_n\rangle = \langle w, v_n\rangle$$

Since the inner product is continuous, if we let $n \to \infty$, we have

$$\langle w, Tu\rangle = \langle S(w), u\rangle = \langle w, v\rangle.$$

Since this holds for all $w \in H$, it follows that $Tu = v$, and so $(u, v) \in Graph(T)$. Hence $Graph(T)$ is closed, and we are done.

# Question 4

## Part (i)

We define the spectrum

$$Spectrum(x) = \{\lambda \in \mathbb{C} \mid x - \lambda 1 \text{ is not invertible }\}.$$

Let $\lambda \in \mathbb{C}$ and $|\lambda| > \|x\|$. Then the element $x/\lambda$ has norm less than one, so by the stated result, $1 - xlambda$ is invertible. Multiplying by $-\lambda$, we see that $x - \lambda 1$ is invertible, and $\lambda \notin Spectrum(x)$. The result now follows.

## Part (ii)

Let $p(z)$ be a complex-valued polynomial. Let $x \in A$. Then

$$Spectrum\, p(x) = p[Spectrum(x)] = \{p(\lambda) \mid \lambda \in Spectrum(x)\}.$$

## Part (iii)

Let $x \in H$. Write $x = v + w$ where $v \in V$ and $w \in V^\perp$. Then $P(x) = v$.

Hence $P^2(x) = P(v) = P(v + 0) = v$, so $P^2 = P$.

## Part (iv)

Let $f(z) = z^2 - 1$. Then $f(P) = P^2 - P = 0$. Hence, by the spectral mapping theorem for polynomials, if $\lambda \in \mathit{Spectrum}(P)$, then $p(\lambda) = \lambda^2 - \lambda = 0$.

It follows that $\lambda = 0$ or $\lambda = 1$, so $\mathit{Spectrum}(P) \subseteq \{0, 1\}$.

Since $V \neq H$, the projection $P$ is not itself invertible. Hence $0 \in \mathit{Spectrum}(P)$.

Now for $v \in V$ and $w \in V^\perp$, we have $(I - P)(v + w) = v + w - v = w$, which is the projection onto $V^\perp$. Since $V \neq \{0\}$, $V^\perp \neq H$, and $I - P$ is not invertible. Hence $1 \in \mathit{Spectrum}(P)$.

We conclude $\mathit{Spectrum}(P) = \{0, 1\}$ as required.

## Part (v)

By the spectral mapping theorem for polynomials and part (iii), we have

$$\mathit{Spectrum}(I + 3P^4) = \{1 + 3\lambda^4 \mid \lambda = 0, 1\} = \{1, 4\}.$$

# Question 5

## Part (i)

We say a linear map $K\colon V \to V$ is compact if the closure $\overline{K[(B(0,1)]}$ is compact, where $B(0,1)$ is the unit ball at 0 in the space $V$.

## Part (ii)

Let $T\colon V \to V$ be a bounded linear map such that the image $T[V]$ is finite-dimensional.

Then $T[V]$ will be a closed subspace of $W$, so $\overline{T[B(0,1)_V]} \subseteq T[V]$. Since $T$ is bounded, if $v \in B(0,1)_V$, then $\|Tv\| \leq \|T\| \cdot \|v\| < \|T\|$. Hence $\|w\| \leq \|T\|$ whenever $w \in \overline{T[B(0,1)]}$. Thus the set $\overline{T[B(0,1)]}$ is a closed bounded subset of a finite-dimensional normed vector space, so it is compact by the Heine-Borel theorem. It follows that the operator $T$ is compact.

## Part (iii)

- Let $(e_n)$ be the standard orthonormal basis for $l^2$. Then $e_n \in \overline{I[B(0,1)]}$. But for $m \neq n$, we have

  $$\|e_m - e_n\|^2 = \langle e_m - e_n, e_m - e_n \rangle = 2$$

  which means $(e_n)$ has no Cauchy subsequence, and hence no convergent subsequence. This means the set $\overline{I[B(0,1)]}$ is not compact, and so that $I$ is not a compact operator.

- A similar argument to the above tells us that $R$ is not compact.

- Let
$$S_n(a_1, a_2, a_3, \ldots) = \left(a_1, \frac{a_2}{2}, \ldots, \frac{a_n}{n}, 0, 0, \ldots\right)$$
Then $S_n$ has finite-dimensional image, and so is compact. Now
$$\|(S - S_n)(a_1, a_2, a_3, \ldots)\|^2 = \frac{|a_{n+1}|^2}{(n+1)^2} + \frac{|a_{n+2}|^2}{(n+2)^2} + \cdots.$$
Hence
$$\|(S - S_n)(a_1, a_2, a_3, \ldots)\|^2 = \frac{1}{(n+1)^2}(|a_1|^2 + |a_2|^2 + \cdots)$$
meaning
$$\|S - S_n\| \leq \frac{1}{(n+1)^2}.$$
So $S$ is the norm limit of the compact operators $(S_n)$, meaning $S$ is itself compact.

## Part (iv)

We say $T: H \to H$ is a Fredholm operator if $\ker(T)$ and $\ker(T^*)$ are finite-dimensional, and the image of $T$ is closed. We define the index

$$Index(T) = \dim \ker T - \dim \ker T^*.$$

## Part (v)

Observe that $\ker(R) = \{0\}$, and

$$R^*(a_1, a_2, a_3, \ldots) = (a_2, a_3, a_4, \ldots).$$

so $\dim(\ker R^*) = 1$. The operator $R$ has closed image, so it is Fredholm, and $Index(R) = -1$.

By the above, the operator $S$ is compact. Hence, by standard results on Fredholm operators, $R + S$ is Fredholm, and

$$Index(R + S) = Index(R) = -1.$$