Question 1

Part (i)

- Suppose $\|(a_n)\| = 0$. Then $\sup\{|a_i| \mid i \in \mathbb{N}\} = 0$. But $|a_i| \geq 0$ for all i, so it follows that $a_i = 0$ for all i, that is to say $(a_n) = 0$.
- Let $(a_n) \in c_0$ and $\alpha \in \mathbb{C}$. Then $|\alpha a_i| = |\alpha| \cdot |\alpha_i|$ for each i, so

  $$\|\alpha(a_n)\| = \|(\alpha a_n)\| = \sup\{|\alpha|\cdot|a_i| \mid i \in \mathbb{N}\} = |\alpha| \sup\{|a_i| \mid i \in \mathbb{N}\} = |\alpha|\cdot\|(a_n)\|.$$

- Let $(a_n), (b_n) \in c_0$. Then

  $$|(a_n)|+(b_n)| = \sup\{|a_i+b_i| \mid i \in \mathbb{N}\} \leq \sup\{|a_i|+|b_j| \mid i,j \in \mathbb{N}\} = \|(a_n)\|+\|(b_n)\|.$$

Part (ii)

We call V a Banach space if it is complete, that is to say every Cauchy sequences in V converges to an element of V.

Part (iii)

Let $(a_n^{(k)})_{k=1}^{\infty}$ be a Cauchy sequence in c_0. Fix $i \in \mathbb{N}$. Then for any k, l

$$|a_i^{(k)} - a_i^{(l)}| \leq \|(a_n^{(k)}) - (a_n^{(l)})\|$$

so it follows that $(a_i^{(k)})_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$.

But the complex numbers $\mathbb{C}$ are complete, so the sequence $(a_i^{(k)})_{k=1}^{\infty}$ converges to some limit, a_i. We claim that $(a_n) \in c_0$, and $\|(a_n^{(k)} - a_n\| \to 0$ as $n \to \infty$. We will prove the second of these first.

Let $\varepsilon > 0$. Since the sequence $(a_n^{(k)})_{k=1}^{\infty}$ is Cauchy, for each $i \in \mathbb{N}$, it follows that we have N such that

$$|a_i^{(k)} - a_i^{(l)}| \leq \|(a_n^{(k)}) - (a_n^{(l)}) < \frac{\varepsilon}{2}$$

whenever $k, l \geq N$.

Thus is we let $l \to \infty$, we see

$$|a_i^{(k)} - a_i| \leq \|(a_n^{(k)}) - (a_n)\| \leq \frac{\varepsilon}{2} < \varepsilon$$

whenever $k \geq N$. So $\|(a_n^{(k)} - a_n\| \to 0$ as $n \to \infty$ as required.

Finally, we need to check that $(a_n) \in c_0$. Again, let $\varepsilon > 0$. By the above, pick K such that $\|(a_n^{(K)}) - (a_n)\| < \frac{\varepsilon}{2}$. Since $(a_n^{(K)}) \in c_0$ we have $N \in \mathbb{N}$ such that $|a_n^{(K)}| < \frac{\varepsilon}{2}$ whenever $n \geq N$.

Now let $n \geq N$. Then

$$|a_n| \leq |a_n - a_n^{(K)}| + |a_n^{(K)}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

so $a_n \to 0$ as $n \to \infty$, and $(a_n) \in c_0$ as we need.

Part (iv)

Let V be a Banach space. We call $A \subseteq V$ *closed* if for any sequence (a_n) in A which converges to a limit $v \in V$, we have $v \in A$.

Part (v)

(a) Any sequence in c_0 that converges in l^∞ is Cauchy. By the above, Cauchy sequences in c_0 have limits in c_0. Therefore c_0 is closed.

(b) Define $v_n \in c_{00}$ by

$$v_n = \left(1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0, \dots\right).$$

Let

$$v = \left(1, \frac{1}{2}, \frac{1}{3}, \dots\right).$$

Observe $\|v_n - v\| = \frac{1}{n+1}$, so the sequence (v_n) has limit v. But $v \notin c_{00}$. So c_{00} is not closed.

(c) The space c_N is a finite-dimensional subspace of l^∞, and is therefore closed.

Question 2

Part (i)

(a) Call T a bounded linear map if there is a constant $M \geq 0$ such that $\|Tx\| \leq \|x\|$ for all $x \in V$. We define the norm

$$\|T\| = \sup_{x \in V \setminus \{0\}} \frac{\|Tx\|}{\|x\|}$$

(b) We call T *open* if it maps open sets to open sets. The open mapping theorem asserts that a surjective bounded linear map between Banach spaces is open.

(c) Let c_{00} be the set of bounded sequences (a_n), which are eventually zero. Define $T\colon c_{00} \to c_{00}$ by $T((a_n)) = (a_n/n)$.

Then T is certainly a surjective bounded linear map. But $T^{-1}((a_n)) = (na_n)$ which does not define a bounded linear map. It follows that T^{-1} is not continuous. By the formulation of continuity in terms of open sets, T is not open.

Part (ii)

(a) We define V^* to be the vector space of bounded linear maps $V \to \mathbb{R}$, with norm as defined above. We want to show that V^* is complete.

Let (f_n) be a Cauchy sequence in V^*. Let $x \in V$. If $x = 0$, then $f_n(x) = 0$ for all n, and $f_n(x) \to 0$ as $n \to \infty$.

So suppose $x \neq 0$. Let $\varepsilon > 0$. Then, as (f_n) is Cauchy, we have $N \in \mathbb{N}$ such that $\|f_m - f_n\| < \varepsilon/\|x\|$ whenever $m, n \geq N$.

Hence, for $m, n \geq N$, we have

$$|f_m(x) - f_n(x)| \leq \|f_m - f_n\| \cdot \|x\| < \varepsilon.$$

So the sequence $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$, for any $x \in V$. Cauchy sequences converge in $\mathbb{R}$, so let $f(x)$ be the limit as $n \to \infty$.

We claim that $f \in V^*$, and $\|f_n - f\| \to 0$ as $n \to \infty$.

Since each f_n is linear, the algebra of limits tells us that f is linear.

Observe

$$|\|f_m\| - \|f_n\|| \leq \|f_m - f_n\|$$

for all $m, n \in \mathbb{N}$. Thus, since the sequence (f_n) is Cauchy, the sequence $(\|f_n\|)$ is a Cauchy sequence in $\mathbb{R}$. Let $\|f_n\| \to C$ as $n \to \infty$.

Then for any $x \in V$, we have

$$\lim_{n \to \infty} \sup_{x \in V \setminus \{0\}} \frac{|f_n(x)|}{\|x\|} = C$$

which means, when we rearrage

$$\|f\| = \sup_{x \in V \setminus \{0\}} \frac{|f(x)|}{\|x\|} = C$$

In particular, f is bounded. Now let $\varepsilon > 0$. Since (f_n) is Cauchy, we have N such that

$$\|f_m(x) - f_n(x)\| \leq \frac{\varepsilon}{2}\|x\|$$

for all $x \in V$, whenever $m, n \geq N$.

Let $m \to \infty$. Then

$$\|f(x) - f_n(x)\| \leq \frac{\varepsilon}{2}\|x\|$$

for all $x \in V$, whenever $m, n \geq N$. It follows that $\|f_n - f\| < \varepsilon$ whenever $n \geq N$, and we are done.

(b) Let W be a subspace of a normed vector space V. Let $f \in W^*$. Then we have $F \in V^*$ such that $F|_W = f$ and $\|F\| = \|f\|$.

(c) Let $v \in V$. Observe

$$\|\tau(v)(f)\| = \|f(v)\| \leq \|f\| \cdot \|v\|$$

for all $f \in V^*$.

It follows that $\|\tau(v) \leq \|v\|$.

If $v = 0$, then $\tau(v) = 0$, so $\|\tau(v)\| = \|v\|$. So suppose $\tau(v) \neq 0$.

Let W be the one-dimensional vector space spanned by the vector v. Define $f\colon W \to \mathbb{K}$ by

$$f(\alpha v) = \alpha \|v\| \qquad \alpha \in \mathbb{K}.$$

Then by the Hahn-Banach theorem, we have $F \in V^*$ such that $\|F\| = \|f\|$, and $F(v) = f(v) = 1$.

We see that

$$\|F\| = \|f\| = \sup_{\alpha \neq 0} \frac{|f(\alpha v)|}{|\alpha| \cdot \|v\|} = 1.$$

Hence

$$\|\tau(v)\| \geq |\tau(v)(F)| = |F(v)| = \|v\|.$$

Combining the two inequalities, we are done.

Question 3

Part (i)

Let X be a compact metric space. Let A be a unital subalgebra of $C(X)$ which separates points. Then A is dense in $C(X)$.

Part (ii)

Let $m > n$. The formulae

$$\cos((m+n)x) = \cos(mx)\cos(nx) - \sin(mx)\sin(nx)$$

$$\cos((m-n)x) = \cos(mx)\cos(nx) + \sin(mx)\sin(nx)$$

let us write

$$\cos(mx)\cos(nx) = \frac{1}{2}(\cos((m+n)x) + \cos((m-n)x)$$

that is as as a linear combination of $\cos(mx)$ and $\cos(nx)$. It follows that A is an algebra.

Allowing $n = 0$, we have that $1 \in A$, so A is a unital algebra.

If $x, y \in [0, \pi]$, then for $x \neq y$ we have $\cos x \neq \cos y$. Hence A separates points.

By the Stone-Weierstrass theorem, A is dense in $C[0, \pi]$.

Part (iii)

Let $M \subseteq (C[0, \pi], \| - \|_\infty)$ be dense, where $\| - \|_\infty$ is the supremum norm. Let $\| - \|$ be the Hilbert space norm on $L^2[0, \pi]$.

Let $f, g \in C[0, \pi]$. Then

$$\|f - g\|^2 = \int_0^\pi (f-g)^2 \leq \pi \|f - g\|_\infty^2 \quad \Rightarrow \quad \|f - g\| \leq \sqrt{\pi}\|f - g\|_\infty.$$

It follows that M is also dense in the space $(C[0, \pi], \|-\|)$. But $(C[0, \pi], \|-\|)$ is itself dense in $L^2[0, \pi]$. So M is also dense in $L^2[0, \pi]$.

Part (iv)

By above, the span of the set $\{e_0, e_1, e_2, \ldots\}$ is dense in $L^2[0, \pi]$. We must check that the set is orthonormal. Let $m > n > 0$. Then

$$\langle e_m, e_n \rangle = \frac{2}{\pi}\int_0^\pi \cos(mx)\cos(nx)\; dx = \frac{1}{\pi}\int_0^\pi \cos((m+n)x)\; dx + \frac{1}{\pi}\int_0^\pi \cos((m-n)x)\; dx = 0.$$

Let $m > 0$. Then

$$\langle e_m, e_0 \rangle = \frac{\sqrt{2}}{\pi}\int_0^\pi \cos(mx)\; dx = 0$$

Observe

$$\|e_0\|^2 = \frac{1}{\pi}\int_0^\pi \, dx = 1.$$

Finally, for $m > 0$

$$\|e_m\|^2 = \frac{2}{\pi}\int_0^\pi \cos^2(mx)\; dx = 1.$$

Part (v)

We have

$$\alpha_n = \langle e_n, f\rangle$$

so for $n = 0$:

$$\alpha_0 = \frac{1}{\sqrt{\pi}}\int_0^\pi \sin x\; dx = \frac{2}{\sqrt{\pi}}$$

For $n > 0$:

$$\alpha_n = \sqrt{\frac{2}{\pi}}\int_0^\pi \cos(nx)\sin x\; dx$$

The formulae

$$\sin((m+1)x) = \sin(mx)\cos(x) + \cos(mx)\sin(x)$$

$$\sin((m-1)x) = (mx)\cos(x) - \cos(mx)\sin(x)$$

let us write

$$\cos(mx)\sin(x) = \frac{1}{2}(\sin((m+1)x) - \sin((m-1)x)$$

so

$$\alpha_n = \frac{1}{\sqrt{2\pi}}\int_0^\pi \sin((n+1)x) - \sin((n-1)x)\; dx$$

that is

$$\alpha_n = \sqrt{\frac{2}{\pi}}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \frac{2\sqrt{2}}{\sqrt{\pi}(n^2-1)}.$$

By Parseval's law

$$\sum_{n=0}^{\infty} |\alpha_n|^2 = \|f\|^2 = \int_0^\pi \sin^2 x\; dx = \frac{1}{2}.$$

Question 4

Part (i)

(a)

$$Spectrum(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \text{ is not } invertible\}.$$

(b) Let $S_n = 1 + x + \cdots + x^n$.

Observe, for $n > m$

$$\|S_n - S_m\| = \|x^{m+1} + \cdots + x^n\| \leq \frac{\|x\|^{m+1}}{1 - \|x\|} \to 0$$

as $m, n \to \infty$, since $\|x\| < 1$.

So (S_n) is a Cauchy sequence, meaning it converges in norm to some $S \in A$ by completeness.

Now

$$(1-x)S_n = (1 + x + x^2 \cdots + x^n) - (x + x^2 + \cdots + x^n + x^{n+1}) = 1 - x^{n+1}$$

so

$$1 - (1-x)S_n\| = \|x^{n+1}\| \leq \|x\|^{n+1}.$$

Let $n \to \infty$. Since $\|x\| < 1$, we see $\|(1-x)S\| = 0$, and so $(1-x)S = 0$.

In the same way, $S(1-x) = 0$, so $1 - x$ is invertible, with $(1-x)^{-1} = S$.

(c) We call U *unitary* if $U^*U = UU^* = I$.

Observe that if U is unitary, then for any $v \in V$,

$$\|Uv\|^2 = \langle Uv, Uv\rangle = \langle U^*Uv, v\rangle = \langle v, v\rangle = \|v\|^2.$$

In particular, $\|U\| = 1$.

Let $\lambda \in \mathbb{C}$ and suppose $\lambda > 1$. Then $\|U/\lambda\| < 1$, so $I - U\lambda$ is invertible by part (b). Multiplying by $-\lambda$, we see that $U - \lambda I$ is invertible, so $\lambda \notin Spectrum(U)$.

In the same way, if $|\lambda| < 1$, then $I - \lambda U^*$ is invertible, and multiplying by U, we see that $U - \lambda I$ is invertible, so $\lambda \notin Spectrum(U)$.

Therefore if $\lambda \in Spectrum(U)$, then $|\lambda| = 1$.

Part (ii)

(a) Observe

$$\left\| \frac{x^n}{n!} \right\| \leq \frac{\|x\|^n}{n!}$$

and the series $\sum_{n=0}^{\infty} \frac{\|x\|^n}{n!}$ converges.

So the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges absolutely, and therefore (since A is a Banach space) also converges to an element $e^x \in A$.

Now

$$\exp(x)\exp(y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!}$$

By the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k y^{n-k}$$

Therefore

$$\exp(x)\exp(y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(x+y)^n}{n!} = \exp(x+y)$$

(b) Observe $(iT)^* = -iT$ as T is self-adjoint. So

$$\exp(iT)^* = \sum_{n=0}^{\infty} \frac{((iT)^*)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-iT)^n}{n!} = \exp(-iT).$$

By the above

$$\exp(iT)\exp(-iT) = \exp(-iT)\exp(iT) = \exp(0) = I.$$

Therefoe, by definition, $\exp(iT)$ is unitary.

Question 5

Part (i)

If V and W are normed vector spaces, we call a linear map $T\colon V \to W$ *compact* if the image $T[B(0,1)]$ has compact closure.

Part (ii)

Let (x_n) be a bounded sequence in V. Say $\|x_n\| < M$ for all n. Then the sequence (x_n/M) is contained in the unit ball $B(0,1)$ in V. Hence the sequence $(T(x_n)/M)$ is contained in the compact space $\overline{T[B(0,1)]}$, and so has a convergent subsequence, $T(x_{n_k})/M$.

It follows that the sequence (Tx_n) has a convergent subsequence (Tx_{n_k}).

Part (iii)

Let $T\colon V \to W$ be a bounded linear map with finite-dimensional image. Then, if $v \in B(0,1)$, we have $\|v\| < 1$, so $\|Tv\| < \|T\|$. It follows that the image $T[B(0,1)]$ is bounded, so the closure $\overline{T[B(0,1)]}$ is also bounded. So $\overline{T[B(0,1)]}$ is a closed bounded subset of a finite-dimensional normed vector space. By the Heine-Borel theorem, this set must be compact, meaning T is a compact operator.

Part (iv)

Observe

$$\|Tx_m - Tx_n\|^2 = \langle Tx_m - Tx_n, Tx_m - Tx_n\rangle = \langle T^*Tx_m - T^*Tx_n, x_m - x_n\rangle$$

Since the sequence (x_n) is bounded, we have a constance $C \geq 0$ such that $\|x_n\| \leq C$ for all n. By the triangle inequality, $\|x_m - x_n\| \leq 2C$ for all m and n. Hence, by the Cauchy-Schwarz inequality

$$\|Tx_m - Tx_n\|^2 \leq 2C\|T^*Tx_m - T^*x_n\|.$$

Since the sequence (T^*Tx_n) converges, it must be a Cauchy sequence.

Let $\varepsilon > 0$. Then we have N such that if $m, n \geq N$, then $\|T^*Tx_m - T^*x_n\| < \varepsilon^2/2C$. By the above, it follows that $\|Tx_m - Tx_n\| < \varepsilon$ for $m, n \geq N$, so the sequence (Tx_n) is Cauchy as desired.

Part (v)

Let (x_n) be a bounded sequence in H. Then as K^*K is compact, (K^*Kx_n) has a convergent subsequence $(K^*Kx_{n_k})$. By the above, the sequence (Kx_{n_k}) is Cauchy, and therefore converges.

Thus (Kx_n) has a convergent subsequence, and therefore K is compact.

Part (vi)

Let K be compact. The product of a compact operator with another operator is compact. Therefore K^*K is compact, and the converse holds.