Functional Analysis: Semester 2 Chapter 5 Solutions

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1. Let $x \in T[H]^\perp$. Then for any $y \in H$ we have
$$\langle T^* x, y\rangle = \langle x, Ty\rangle = 0.$$
Hence, taking $y = T^* x$, we see that $T^* x = 0$, and $x \in \ker T^*$.
Conversely, let $x \in \ker T^*$. Then for any $y \in H$ we have
$$\langle x, Ty\rangle = \langle T^* x, y\rangle = 0$$
so $x \in T[H]^\perp$.
Therefore $\ker T^* = T[H]^\perp$ as required.

2. (a) Let $\alpha\colon V \to \mathbb{F}$ be a linear map. Suppose that α is continuous. Then the inverse image under α of an open set is open, so the inverse image under α of a closed set is closed. Thus
$$\ker \alpha = \alpha^{-1}[\{0\}]$$
is closed.
Conversely, suppose that $\ker \alpha$ is closed. If $\alpha = 0$,. then α is clearly continuous. So suppose that $\alpha \neq 0$.
So the complement of $\ker \alpha$ is non-empty and open. It follows that we have $x \in V$ and $\delta > 0$ such that $B(x, \delta) \cap \ker \alpha = \emptyset$.
We claim that $\alpha[B(0, \delta)]$ is bounded. Suppose otherwise. Then we have a sequence (x_n) in $B(0, \delta)$ such that $|\alpha(x_n)| \to \infty$ as $n \to \infty$. In particular, there exists $y \in B(0, \delta)$ such that $|\alpha(y)| > |\alpha(x)|$.
Set
$$z = \frac{-\alpha(x) y}{\alpha(y)}.$$
Then $z \in B(0, \delta)$, and $\alpha(z) = -\alpha(x)$. So $z + x \in B(x, \delta)$, and $\alpha(z + x) = 0$, so $z + x \in \ker \alpha$. But this statement is a contradiction; we had that $B(x, \delta) \cap \ker \alpha = \emptyset$.
So the set $\alpha[B(0, \delta)]$ must be bounded. So we have a constant M such that $|\alpha(x)| \leq M$ whenever $x \in B(0, \delta)$.

Now, for any $x \in X$, we have $\frac{\delta x}{2\|x\|} \in B(0,\delta)$, so

$$\frac{\delta}{2\|x\|}|\alpha(x)| \leq M.$$

Rearranging, we see that $|\alpha(x)| \leq \frac{2M}{\delta}\|x\|$ for all $x \in V$, and so α is a bounded linear functional, and hence continuous.

(b) We prove the result when $\dim B = 1$ and $A \cap B = \{0\}$; the general result then follows by induction.
Let $B = Span\{e\}$, where $e \notin A$. Then $A + B = A \oplus Span\{e\}$.
Define $\alpha\colon A+B \to \mathbb{F}$ by the formula $\alpha(a+\lambda e) = \lambda$. Then α is linear, and $\ker\alpha = A$, which is closed, so α is continuous by the above.
Let $v \in \overline{A+B}$. Then we have a sequence (v_n) in $A+B$ such that $\|v_n - v\| \to 0$ as $n \to \infty$.
Write $v_n = a_n + \lambda_n e$. By continuity of the map α, the sequence (λ_n) is Cauchy, and so converges to a limit $\lambda \in \mathbb{F}$. Write $a = v - \lambda e$. Then $\|a_n - a\| \to 0$ as $n \to \infty$.
Thus $a \in \overline{A} = A$, and $v = a + \lambda e \in A + B$. Hence $A + B$ is closed as required.

3. (a) Let $a = (a_1, a_2, a_3, \ldots) \in l^2$.
The map T is clearly linear, and

$$\|T(a)\|^2 = \frac{|a_1|^2}{1} + \frac{|a_2|^2}{4} + \frac{|a_3|^2}{9} + \cdots \leq |a_1|^2 + |a_2|^2 + |a_3|^2 + \cdots = \|a\|^2$$

so the map T is bounded.
Let

$$x_n = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{2}, 0, 0, \ldots\right) \qquad x = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right).$$

Set $y_n = (1,1,1,\ldots,1,0,0,0,\ldots)$, so that $y_n \in l^2$ and $T(y_n) = x_n$.
Observe $\|x - x_n\| \to 0$ as $n \to \infty$ so $x \in \overline{T[l^2]}$.
Suppose $x = Ty$. Then we must have, by definition of the map T

$$y = (1,1,1,\ldots).$$

But $y \notin l^2$ (the norm would be infinite). So $x \neq Ty$ for all $y \in l^2$, that is $x \notin T[l^2]$. So $T[l^2]$ is not closed.

(b) Let $H = l^2 \oplus l^2$. Since the above map T is not continuous, the graph $A = Gr(T) \subseteq H$ is closed.
Let $B = l^2 \oplus 0$. Then B is closed. But

$$A + B = l^2 \oplus T[l^2]$$

which is not closed by the above.

4. (a) Observe
$$(T_1 \oplus T_2)[H] = T_1[H] \oplus T_2[H]$$
which is closed, and
$$\ker(T_1 \oplus T_2) = \ker T_1 \oplus \ker T_2 \qquad \ker((T_1 \oplus T_2)^*) = \ker T_1^* \oplus \ker T_2^*.$$
The result now follows.

(b) Since the operator S is invertible, and $T[H] \subseteq H$ is closed, so is $ST[H]$. Certainly, $\ker(ST) = \ker T$, and, as S^* is also invertible,
$$\dim \ker(T^* S^*) = \dim \ker T^*.$$
It follows that the composite ST is Fredholm, with $Ind(ST) = Ind(T)$.

5. (a) We know by proposition 5.28 that $Ind(T_f) = -3$.

(b) Let $\alpha(s) = \exp(ie^{is})$, $s \in [0, 2\pi]$. Then, by the Toeplitz index theorem
$$Ind(T_f) = -\frac{1}{2\pi i} \oint_\alpha \frac{1}{z}\, dz.$$
We have $\alpha'(s) = -e^{is} \exp(ie^{is})$, so
$$Ind(T_f) = -\frac{1}{2\pi i} \int_0^{2\pi} \frac{-e^{is} \exp(ie^{is})}{\exp(ie^{is})}\, ds = \frac{1}{2\pi i} \int_0^{2\pi i} e^{is}\, ds = 0.$$

(c) This question needs some complex analysis covered in the functional analysis course, but covered in prerequisite courses.
Let $\alpha(s) = \cos(2e^{is})$. Then $\alpha'(s) = -2ie^{is} \sin(2e^{is})$, so as above, by the Toeplitz index theorem
$$Ind(T_f) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{2ie^{is} \sin(2e^{is})}{\cos(2e^{is})}\, ds = \frac{1}{2\pi} \int_0^{2\pi} 2ie^{is} \tan(2e^{is})\, ds$$
Let $\beta(s) = 2e^{is}$. Then $\beta'(s) = 2ie^{is}$, so
$$Ind(T_f) = \frac{1}{2\pi i} \oint_\beta \tan z\, dz.$$
Now,
$$\tan z = \frac{\sin z}{\cos z} = \frac{g(z)}{h(z)}$$
is holomorphic apart from simple poles where $\cos z = 0$, that is to say $z = k\pi + \frac{\pi}{2}$, where $k \in \mathbb{Z}$.

Precisely two of these poles, namely those at $z = \pm\frac{\pi}{2}$, lie inside the contour β, which is a circle with radius 2. At these poles we have residues

$$\mathrm{Res}(\tan z; \frac{\pi}{2}) = \frac{g(\pi/2)}{h'(\pi/2)} = -1 \qquad \mathrm{Res}(\tan z; -\frac{\pi}{2}) = \frac{g(-\pi/2)}{h'(-\pi/2)} = -1.$$

So by Cauchy's residue theorem

$$Ind(T_f) = \frac{1}{2\pi i}\oint_\beta \tan z \; dz = -2.$$