Functional Analysis: Semester 2 Chapter 4 Solutions

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1. (a) Let $T\colon H \to H$ be unitary. Then $T^* = T^{-1}$, so T is an isomorphism. Let $x \in H$. Then

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle = \langle x, x\rangle = \|x\|^2$$

so T is an isometry.

Conversely, let T be an isometric isomorphism. Then $\|Tv\|^2 = \|v\|^2$ for all $v \in H$.

Let $x, y \in H$. Then if H is a real Hilbert space,

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

and if H is a complex Hilbert space,

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) + \frac{i}{4}(\|x+iy\|^2 - \|x-iy\|^2).$$

Hence $\langle Tx, Ty\rangle = \langle x, y\rangle$. It follows that $\langle T^*Tx, y\rangle = \langle x, y\rangle$. Since this identity holds for all $x, y \in H$, we have $T^*T = I$.

Since the map T is an isomorphism, it is invertible. By the above, we must have $T^{-1} = T^*$, meaning T is unitary.

(b) The right shift operator, R, on l^2 is defined by the formula

$$R(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, a_3, \ldots).$$

Clearly R is an isometry. But R is not surjective, and therefore not an invertible; it therefore cannot be a unitary.

2. Let $T\colon H \to H$ be self-adjoint. Let $v \in H$. Then

$$\langle v, Tv\rangle = \langle Tv, v\rangle = \overline{\langle v, Tv\rangle}$$

so $\langle v, Tv\rangle \in \mathbb{R}$.

Conversely, suppose $\langle v, Tv\rangle \in \mathbb{R}$ for all $v \in H$. Let $x, y \in H$. We need to show

$$\langle Tx, y\rangle = \langle x, Ty\rangle.$$

Let $\alpha \in \mathbb{C}$. Then

$$\langle x + \alpha y, T(x + \alpha y)\rangle = \langle x, Tx\rangle + \overline{\alpha}\langle y, Tx\rangle + \alpha\langle x, Ty\rangle + |\alpha|^2\langle y, Ty\rangle \in \mathbb{R}.$$

Thus

$$\overline{\alpha}\overline{\langle Tx, y\rangle} + \alpha\langle x, Ty\rangle \in \mathbb{R}.$$

Set $\alpha = 1$. Then

$$\Im(\overline{\langle Tx, y\rangle} + \langle x, Ty\rangle) = 0$$

that is

$$\Im(\langle Tx, y\rangle) = \Im(\langle x, Ty\rangle).$$

Taking $\alpha = i$, we similarly see that $\Re(\langle Tx, y\rangle) = \Re(\langle x, Ty\rangle)$. Therefore

$$\langle Tx, y\rangle = \langle x, Ty\rangle$$

and we are done.

3. Let $T: V \to W$ be compact. Let (x_n) be a bounded sequence in V. Let $B = \{x_n \mid n \in \mathbb{N}\}$.

   Then the image $T[B]$ has compact closure. Thus every sequence in B has a convergent subsequence. In particular, the sequence (Tx_n) has a convergent subsequence.

   Conversely, suppose that for every bounded sequence (x_n), the image (Tx_n) has a convergent subsequence. Let (y_n) be a sequence in $T[B_V(0,1)]$. Then $y_n = Tx_n$, where (x_n) is bounded. Hence (y_n) has a convergent subsequence, with limit in the closure $\overline{T[B_V(0,1)]}$. It follows by definition of compactness that the closure $\overline{T[B_V(0,1)]}$ is compact.

4. Define $T_N: l^2 \to l^2$ by

$$T_N(a_1, a_2, a_3, \ldots) = (a_1, \frac{a_2}{2}, \ldots, \frac{a_N}{N}, 0, 0, \ldots).$$

   Then

$$\|T(a_1, a_2, a_3, \ldots) - T_N(a_1, a_2, a_3, \ldots)\|^2 = \sum_{n=N+1}^{\infty} \frac{|a_n|^2}{n^2} \leq \frac{1}{N+1}\sum_{n=1}^{\infty} |a_n|^2 = \frac{1}{N+1}\|(a_1, a_2, a_3, \ldots)\|^2.$$

   Thus $\|T - T_N\| \leq 1/(N+1)$, and so $\|T - T_N\| \to 0$ as $N \to \infty$.

   Now each map T_N has finite rank, and is therefore compact. It follows that T is compact.

   Checking that T is self-adjoint is completely straightforward.

5. Define $T\colon C[0,1] \to C[0,1]$ by

$$(Tf)(x) = xf(x).$$

Then

$$\|Tf\|^2 = \int_0^1 |xf(x)|\ dx \leq \int_0^1 |f(x)|^2\ dx = \|f\|^2.$$

So T is a bounded linear map, and therefore extends to a bounded linear map on the Hilbert space $L^2[0,1]$.

Let $f, g \in C[0,1]$. Then

$$\langle Tf, g\rangle = \int_0^1 x\overline{f(x)}g(x)\ dx = \langle f, Tg\rangle.$$

This equation extends by continuity to $L^2[0,1]$, so the operator T is self-adjoint.

Now, let λ be an eigenvalue of T, and let f be a corresponding eigenvector. Then, at least for f a continuous function, we have

$$xf(x) - \lambda f(x) = 0 \qquad \text{for all } x \in [0,1].$$

Hence $f(x) = 0$ or $x = \lambda$ for any value $\lambda \in [0,1]$. Thus $f(x) = 0$ for all $x \in [0,1]$ except, possibly, for one value.

It follows that $\|f\| = 0$, meaning (even for f not continuous), f corresponds to the point 0 in $L^2[0,1]$. Thus, if $Tf = \lambda f$, then $f = 0$, and the operator T has no eigenvalues.

6. (a) Observe

$$(Af)(s) = \int_0^s t(1-s)f(t)\ dt + \int_s^1 s(1-t)f(t)\ dt.$$

Suppose $Af = \lambda f$, and $\lambda \neq 0$. Then $(Af)(0) = (Af)(1) = 0$, so $f(0) = f(1) = 0$.
Now, let $y = Af$. Then, differentiating the above, we see

$$y''(s) + f(s) = y''(s) + \frac{y(s)}{\lambda} = 0.$$

This equation has general solution

$$y(s) = A\cos(s/\lambda) + B\sin(s/\lambda).$$

But we know that $y(0) = y(1) = 0$. Hence we must have $A = 0$, and $1/\lambda = k\pi$ where $k \in \mathbb{Z}$, $k \neq 0$. It follows that the set of non-zero eigenvalues is

$$\{\frac{1}{k\pi} \mid k \in \mathbb{Z}\backslash\{0\}\}$$

and the eigenvalue $1/k\pi$ has a corresponding eigenvector $\sin(k\pi s)$.

(b) Observe that A is a compact self-adjoint operator. Hence

$$\|A\| = \sup\{|\lambda| \mid \lambda \in Spectrum(A)\} = \frac{1}{\pi}.$$