# Functional Analysis: Semester 2 Chapter 3 Solutions

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1. (a) Let $x \in A$, and $\lambda \in \mathbb{F}$, with $\|x\| < \lambda$. Then $\lambda \neq 0$, and $\|x/\lambda\| < 1$, so by the Carl Neumann criterion, the element $1 - x/\lambda$ is invertible in $A$.
   Let
   $$y = \frac{1}{\lambda}\left(1 - \frac{x}{\lambda}\right).$$
   Observe
   $$(\lambda - x)y = (1 - x/\lambda)(1 - x/\lambda)^{-1} = 1.$$
   Similarly, $y(\lambda - x) = 1$. Thus $\lambda - x$ is invertible, with inverse $y$.

   (b) Write $GL(A)$ to denote the set of invertible elements of $A$. Let $x \in GL(A)$. Let $y \in A$, and set $z = x - y$, so that $y = x - z$.
   Hence $yx^{-1} = 1 - zx^{-1}$.
   Certainly $x^{-1} \neq 0$, so $\|x^{-1}\| > 0$.
   Let $\|z\| < 1/\|x^{-1}$. Then $\|zx^{-1}\| < 1$, and by the Carl Neumann criterion, $zx^{-1}$ is invertible, meaning $y = x - z$ is invertible (and so in $GL(A)$) whenever $z$ is sufficiently small.
   Thus the set $GL(A)$ is an open set.

2. (a) Let $v = (a_1, a_2, a_3, \ldots) \in l^2$. Suppose $Rv = \lambda v$, where $\lambda \in \mathbb{C}$. Then
   $$\begin{array}{rcl} 0 & = & \lambda a_1 \\ a_1 & = & \lambda a_2 \\ a_2 & = & \lambda a_3 \\ \vdots & & \vdots \end{array}$$
   If $\lambda = 0$, we get $a_i = 0$ for all $i$ from the above, so $v = 0$ and $\lambda$ is not an eigenvalue.
   If $\lambda \neq 0$, the above tells us that $a_1 = 0$. Iterating, we see again that $a_i = 0$ for all $i$. Again we conclude that $v = 0$ and so $\lambda$ is not an eigenvalue.
   So the set of eigenvalues of $R$ is the empty set.

(b) Let $\lambda \in \mathbb{C}$ and $v = (a_1, a_2, \ldots)$. Suppose $(R - \lambda I)v = (b_1, b_2, \ldots)$. Then

$$\begin{array}{rcl} b_1 &=& -\lambda a_1 \\ b_2 &=& a_1 - \lambda a_2 \\ b_3 &=& a_2 - \lambda a_3 \\ \vdots && \vdots \end{array}$$

Thus, if $\lambda \neq 0$, then

$$\begin{array}{rcl} a_1 &=& -\lambda^{-1} b_1 \\ a_2 &=& -\lambda^{-1} b_2 - \lambda^{-2} b_1 \\ a_3 &=& -\lambda^{-1} b_3 - \lambda^{-2} b_2 - \lambda^{-3} b_1 \\ \vdots && \vdots \end{array}$$

Let $0 < |\lambda| \leq 1$. Suppose $R - \lambda I$ is invertible. Let $v = (1, 0, 0, \ldots)$. Then by the above

$$(R - \lambda I)^{-1}(v) = (\lambda^{-1}, \lambda^{-2}, \lambda^{-3}, \ldots)$$

and $(R - \lambda I)^{-1}(v) \notin l^2$. But this statement is a blatant contradiction, so $R - \lambda I$ is not invertible, and $\lambda \in Spectrum(R)$.

Observe that $R = R - 0I$ is not surjective, and therefore not invertible. Hence $0 \in Spectrum(R)$.

It is easy to see that $\|R\| = 1$. We know from the lectures that if $|\lambda| > \|R\| = 1$, then $\lambda \notin Spectrum(R)$.

We conclude then that

$$Spectrum(R) = \{\lambda \in \mathbb{C} \mid \|\lambda\| \leq 1\} = \overline{B_{\mathbb{C}}(0, 1)}.$$

(c) Observe that $T = I - 2R^2$.

Hence by the spectral mapping theorem

$$Spectrum(T) = \{1 - 2\lambda^2 \mid \lambda \in Spectrum(R)\} = \overline{B_{\mathbb{C}}(1, 2)}.$$