Functional Analysis: Semester 2 Chapter 2 Solutions

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1. (a) Observe $\gamma_k'(t) = 2\pi i k \exp(2\pi i k t)$, so

$$\oint_{\gamma_k} f(z)\ dz = 2\pi i k \int_0^1 e^{2\pi i k t} f(e^{2\pi i k t}\ dt.$$

Let $s = kt$. Using the above substitutiuon rule, we see

$$\oint_{\gamma_k} f(z)\, dz = 2\pi i \int_0^k e^{2\pi i s} f(e^{2\pi i s}\ ds = 2\pi i k \int_0^1 e^{2\pi i s} f(e^{2\pi i s}\ ds = \oint_{\gamma_1} f(z)\, dz.$$

(b) Let $|a| < 1$. Let $\delta_k(t) = a + (1-a)\exp(2\pi i k t)$. Then $\delta_k(0) = \delta_k(1) = 1$.

Since $|a| < 1$, we have a homotopy $H\colon [0,1] \times [0,1] \to \mathbb{C}\backslash\{a\}$ between γ_k and δ_k defined by the formula

$$H(t,s) = (1-s)\gamma_k(t) + s\delta_k(t).$$

Hence $Wind(\gamma_k; a) = Wind(\delta_k; a)$, and

$$Wind(\delta_k; a) = \frac{1}{2\pi i} \oint_{\delta_k} \frac{1}{z-a}\ dz.$$

Now $\delta_k'(t) = 2\pi i k(1-a)\exp(2\pi i k t)$, so

$$Wind(\delta_k; a) = \frac{1}{2\pi i} \int_0^1 \frac{2\pi i k(1-a)e^{2\pi i k t}}{(1-a)e^{2\pi i k t}}\ dt = k.$$

Let $|a| > 1$. Let $c\colon [0,1] \to \mathbb{C}$ be the constant map $c(t) = 0$. Since $|a| < 1$, we have a homotopy $H\colon [0,1] \times [0,1] \to \mathbb{C}\backslash\{a\}$ between γ_k and c defined by the formula

$$H(t,s) = (1-s)\gamma_k(t).$$

Hence

$$Wind(\gamma_k; a) = \frac{1}{2\pi i} \oint_{\gamma_k} \frac{1}{z-a}\ dz = 0$$

by Cauchy's theorem.

2. Let $D \subseteq \mathbb{C}$ be open, and let $f\colon D \to \mathbb{C}$ be holomorphic. Let γ be a closed path in D with $Wind(\gamma, w) = 1$. Then f is infinitely differentiable, and

$$f^{(n)}(w) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-w)^{n+1}}\, dz$$

3. (a) Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be the loop $\gamma(t) = a + re^{it}$. Since $w \in B(a, r)$, the loop γ winds once anticlockwise around the point w. Hence, by Cauchy's formula for derivatives

$$|f'(w)| = \left| \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-w)^2}\, dz \right|$$

Hence

$$|f'(w)| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(a+re^{it})|}{|a+re^{it}-w|^2} |ire^{it}|\, dt$$

Now, $|f(w+re^{it})| \leq M$, and $|a+re^{it}-w| \geq r - |w-a|$. Hence, since $|w-a| < r$, we have

$$|f'(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{Mr}{(r-|w-a|)^2}\, dt = \frac{Mr}{(r-|w-a|)^2}$$

and we are done.

(b) Since the function f is bounded, we have a constant $K \geq 0$ such that $|f(z)| \leq K$ for all $z \in \mathbb{C}$. Choose $w \in \mathbb{C}$, and let $r > |w|$. then by the above inequality

$$|f'(w)| \leq \frac{Mr}{(r-|w|)^2}$$

let $r \to \infty$. Then $Mr/(r-|w|)^2 \to 0$. It follows that $f'(w) = 0$. Hence, since the domain $\mathbb{C}$ is certainly connected, the function f is constant.

4. (a) Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ where $n \geq 1$, and $a_n \neq 0$. Write

$$p(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z_n} \right).$$

Certainly

$$\lim_{z\to\infty} \left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z_n} \right) = a_n \qquad \lim_{z\to\infty} z^n = \infty.$$

Hence

$$\lim_{z\to\infty} p(z) = \infty.$$

(b) Suppose $\lim_{z\to\infty} f(z) = \infty$.
Let $\varepsilon > 0$. Then we have $R > 0$ such that $|f(z)| > 1/\varepsilon$ whenever $|z| > R$.
Hence
$$\left|\frac{1}{f(z)}\right| < \varepsilon$$
whenever $|z| > R$.
We see that
$$\lim_{z\to\infty} \frac{1}{f(z)} = 0.$$
The converse is similar.

(c) Suppose that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then the function $z \mapsto \frac{1}{p(z)}$ is a holomorphic function defined on the entire complex plane.
Since p is non-constant, by the above, $\frac{1}{p(z)} \to 0$ as $z \to \infty$.
Hence the function $\frac{1}{p}$ is bounded, and therefore constant by Liouville's theorem. Thus the polynomial p is constant, which is a contradiction.
It follows that there exists $z \in \mathbb{C}$ such that $p(z) = 0$.