Functional Analysis: Chapter 5 Solutions

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1. (a) An infinite set is linearly independent when every finite subset is linearly independent.

   (b) By the above, it suffices to prove the result for finite orthogonal sets. So, let $\{v_1, \ldots, v_n\}$ be orthogonal. Then $v_i \neq 0$ for all i, and $\langle v_i, v_j \rangle = 0$ whenever $i \neq j$.
   Suppose
   $$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0.$$
   Then for each i
   $$\alpha_1 \langle v_i, v_1 \rangle + \cdots \langle v_i, v_n \rangle \alpha_i \langle v_i, v_i \rangle = 0$$
   so $\alpha_i = 0$ as $v_i \neq 0$. Therefore the set $\{v_1, \ldots, v_n\}$ is linearly independent, and we are done.

2. Let H be a Hilbert space. Let $\mathcal{P}$ be the collection of all orthonormal subsets of H, partially ordered by inclusion. Let C be a chain in $\mathcal{P}$. Let
   $$A = \bigcup_{B \in C} B.$$
   We claim that A is linearly orthonormal. To see this, let $x, y \in A$. Suppose $x \in B_1$ and $y \in B_2$ where $B_1, B_2 \in C$. Then either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. But the sets B_1 and B_2 are both orthonormal, so $\langle x, y \rangle = 0$.

   Certainly, $\|x\| = 1$ for all $x \in A$. Thus the set A is orthonormal, and $A \in \mathcal{P}$. Certainly $B \subseteq A$ for all $B \in C$. Thus A is an upper bound for the chain C.

   By Zorn's lemma, it follows that the set $\mathcal{P}$ has a maximal elemt, meaning we have an orthonormal set S such that if S is a subset of an orthonormal set A, then $S = A$.

   We claim now that $\overline{Span(S)} = H$, meaning that S is an orthonormal basis.

   Suppose otherwise. Then $\overline{Span(S)} \neq H$, which means $Span(S)^\perp \neq \{0\}$. Pick $x \in Span(S)^\perp$ such that $\|x\| = 1$. Set $A = S \cup \{x\}$. Then A is an orthonormal set containing S.

   But this statement contradicts the above, and so $\overline{Span(S)} = H$ as required.

3. (a) Certainly $S \subseteq Span(S)$, so $Span(S)^\perp \subseteq S^\perp$. Conversely, let $x \in S^\perp$. Then $\langle x, s\rangle = 0$ for all $s \in S$.
Let $v \in Span(S)^\perp$. Write

$$v = \alpha_1 s_1 + \cdots + \alpha_n s_n \qquad \alpha_i \in \mathbb{F}.$$

Then

$$\langle x, v\rangle = \alpha_1\langle x, s_1\rangle + \cdots + \alpha_n\langle x, s_n\rangle$$

so $x \in Span(S)^\perp$.

(b) Let $S = \{e_n \mid n \in \mathbb{N}\}$. Then we need to show that $\overline{Span(S)} = H$. To do this, it suffices to show that $Span(S)^\perp = \{0\}$, or equivalently by the above, that $S^\perp = \{0\}$.
So, let $x \in S^\perp$. Then $\langle e_n, x\rangle = 0$ for all n. By the stated equation, it follows that $\|x\|^2 = 0$. Thus $x = 0$. We conclude that $S^\perp = \{0\}$ and we are done.

4. (a) Consider the norms

$$\|f\|_\infty = \sup\{|f(t)| \mid t \in [0, \pi]\}$$

and

$$\|f\|_2 = \left(\int_0^\pi |f(t)|^2 \, dt\right)^{\frac{1}{2}}$$

on the space $C[0, \pi]$. Observe

$$(\|f - g\|_2)^2 = int_0^\pi |f(t) - g(t)|^2 \, dt \leq \pi(\|f - g\|_\infty)^2$$

so $\|f - g\|_2 \leq \sqrt{\pi}\|f - g\|_\infty$
Let $S \subseteq C[0, \pi]$ be dense under the norm $\| - \|_\infty$. Let $\varepsilon > 0$. Then for any $h \in L^2[0, \pi]$ we have $g \in C[0, \pi]$ such that $\|g - h\|_2 < \frac{\varepsilon}{2}$, and $f \in S$ such that $\|f - g\|_\infty < \frac{\varepsilon}{2\sqrt{\pi}}$. Hence, by the above $\|f - h\|_2 < \varepsilon$, and S is a dense subset of $L^2[0, \pi]$.
From problem sheet 3, the span of the set of functions f_n is a dense subset of $C[0, \pi]$ under the norm $\| - \|_\infty$. By the above, this set is also a dense subset of $L^2[0, \pi]$.

(b) By the above, we just need to check that the set is orthonormal. Observe

$$\langle e_0, e_0\rangle = \frac{1}{\pi}\int_0^\pi 1 \, dx = 1$$

and for $n \geq 1$

$$\langle e_n, e_n\rangle = \frac{2}{\pi}\int_0^\pi \cos^2(nx) \, dx = \frac{2}{\pi} \times \frac{1}{2}\pi = 1$$

and

$$\langle e_0, e_n\rangle = \sqrt{\frac{2}{\pi}}\int_0^\pi \cos(nx) \, dx = 0$$

Finally

$$cos(mx)\cos(nx) = \frac{1}{2}(\cos((m+n)x) + \cos((m-n)x))$$

so if $m \neq n$ and $m, n \geq 1$

$$\langle e_m, e_n \rangle = \frac{1}{\pi}\int_0^\pi \cos((m+n)x) + \cos((m-n)x)\ dx = 0$$

The result now follows.

(c) We have

$$\alpha_n = \langle f, e_n \rangle = \sqrt{\frac{2}{\pi}}\int_0^\pi \sin x \cos(nx)\ dx$$

Now

$$\sin((n+1)x) - \sin((n-1)x) = 2\cos(nx)\sin x$$

so for $n \geq 2$

$$\alpha_n = \frac{1}{\sqrt{2\pi}}\int_0^\pi \sin((n{+}1)x{-}\sin((n{-}1)x)\ dx = \frac{1}{\sqrt{2\pi}}\left(\frac{2}{n+1} - \frac{2}{n-1}\right) = -\frac{4}{\sqrt{2\pi}(n^2-1)}$$

Similarly,

$$\alpha_1 = \langle f, e_1 \rangle = \sqrt{\frac{2}{\pi}}\int_0^\pi \sin x \cos x\ dx = 0$$

By Parseval's theorem

$$\sum_{n=0}^{\infty} |\alpha_n|^2 = \|f\|^2 = \int_0^\pi \sin^2 x\ dx = \frac{\pi}{2}$$

5. Solution in lecture notes.