Functional Analysis: Chapter 4 Solutions

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1. (a) Observe $\langle x, 0\rangle = 0$ for all $x \in S$. Let $\alpha, \beta \in \mathbb{F}$ and $u, v \in S^\perp$. Let $x \in S$. Then

$$\langle x, \alpha u + \beta v\rangle = \alpha\langle x, u\rangle + \beta\langle x, v\rangle = 0$$

since $u, v \in S^\perp$. Hence $\alpha u + \beta v \in S^\perp$, and $S^\perp$ is a linear subspace of H.

(b) Let $x \in S$ and $v \in S^\perp$. Then

$$lx + v\|^2 = \langle x, x\rangle + \langle x, v\rangle + \langle v, x\rangle + \langle v, v\rangle = \|x\|^2 + \|v\|^2.$$

(c) Suppose

$$v \in \left(\bigcup_{i\in I} S_i\right)^\perp.$$

Then $\langle x, v\rangle = 0$ for all $x \in S_i$, and all $i \in I$. Hence $v \in S_i^\perp$ for all $i \in I$, meaning

$$v \in \bigcap_{i\in I} S_i^\perp.$$

Conversely, suppose

$$v \in \bigcap_{i\in I} S_i^\perp.$$

Then $v \in S_i^\perp$ for all $i \in I$, which tells us that $\langle x, v\rangle = 0$ for all $x \in S_i$ and all $i \in I$. Thus

$$v \in \left(\bigcup_{i\in I} S_i\right)^\perp$$

and we are done.

(d) Let $v \in S^\perp$. Then $\langle x, v\rangle = 0$ for all $x \in S$. Since $S' \subseteq S$, we know in particular that $\langle x, v\rangle = 0$ for all $x \in S'$. So $v \in (S')^\perp$.

(e) Certainly $V \subseteq \overline{V}$, so by the above, $\overline{V}^\perp \subseteq V^\perp$.

So let $w \in V^\perp$; we must prove that $w \in \overline{V}^\perp$, that is that $\langle v, w\rangle = 0$ for all $v \in \overline{V}$.

Consider $v \in \overline{V}$. Let (v_n) be a sequence of vectors in V with norm limit v. Then, since $w \in V^\perp$, we have $\langle v_n, w\rangle = 0$.

By the Cauchy-Schwarz inequality, we have

$$|\langle v_n - v, w\rangle| \leq \|v_n - v\| \cdot \|w\|$$

so $\langle v_n - v, w\rangle \to 0$ as $n \to \infty$.

Thus $\langle v_n, w\rangle \to \langle v, w\rangle$ as $n \to \infty$. But $\langle v_n, w\rangle = 0$ for all n. Therefore $\langle v, w\rangle = 0$, and we are done.

2. Let $v \in T[H]^\perp$. Let $w \in H$. Then

$$\langle T^* v, w\rangle = \langle v, Tw\rangle = 0$$

Since this holds for all $w \in H$, it follows that $T^* v = 0$, and so $v \in \ker T^*$.

Conversely, let $v \in \ker T^*$, and $w \in H$. Then $T^* v = 0$, and so

$$\langle v, Tw\rangle = \langle T^* v, w\rangle = 0$$

Since this holds for all $w \in H$, we see $v \in T[H]^\perp$.

So $T[H]^\perp = \ker T^*$ and we are done.

3. (a) Firstly, observe that for any $v \in H$, the map $J(v)\colon H \to H$ is linear. By the Cauchy-Schwarz inequality

$$|J(v)(x)| = |\langle v, x\rangle| \leq \|v\| \cdot \|x\|$$

for all $x \in H$.

Hence $J(v) \in H^*$, and $\|J(v)\| \leq \|v\|$.

Now observe

$$\frac{|J(v)(v)|}{\|v\|} = \frac{|\langle v, v\rangle|}{\|v\|} = \|v\|.$$

Hence $\|J(v)\| \geq \|v\|$, so, together with the above, $\|J(v)\| = \|v\|$, so the map J is an isometry.

The inner product is conjugate-linear in the first variable. Conjugate-linearity of the map J easily follows.

(b) Since the map J is an isometry, it is injective. We need to show that J is surjective.

Let $f \in H^*$. By the Riesz representation theorem, there is a unique element $R_f \in H$ such that $f(x) = \langle R_f, x\rangle$ for all $x \in H$.

Thus $J(R_f) = f$ and the map J is surjective as required.

(c) By the above, we have conjugate-linear isometric invertible maps

$$J\colon H \to H^* \qquad J^*\colon H^* \to (H^*)^*.$$

So we have an isometric invertible map

$$J^* \circ J\colon H \to (H^*)^*.$$

We need to prove that this map is linear. Let $\alpha, \beta \in \mathbb{F}$ and $v, w \in H$. Then

$$J^* \circ J(\alpha x + \beta y) = J^*(\overline{\alpha} J(x) + \overline{\beta} J(y)) = \alpha J^* \circ J(x) + \beta J^* \circ J(y).$$

This completes the proof.

4.

$$A^*(a_1, a_2, \ldots) = \left(\frac{a_2}{2}, \frac{a_3}{3}, \frac{a_4}{4}, \ldots\right)$$

and

$$B^*(b_1, b_2, \ldots) = \left(\frac{b_2}{2}, b_1, \frac{b_4}{2}, b_3, \ldots\right)$$