Functional Analysis: Problem Sheet 3

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1. (a) Let $v \in V$, $v \neq 0$. Define $f\colon Span\{v\} \to \mathbb{F}$ by $f(\alpha v) = \alpha$. Then $f \in V^*$. By the Hahn-Banach theorem, we have $F \in V^*$ such that $F|_{Span(v)} = f$ and $\|F\| = \|f\|$. In particular, $F(v) = 1$, so $F \neq 0$. Thus $V^* \neq \{0\}$.

   (b) Suppose $x \neq y$. If x and y are not linearly independent, then either $x = \alpha y$ or $y = \alpha x$ for some $\alpha \in \mathbb{R}$, $\alpha \neq 1$. Suppose that $y = \alpha x$. Then $x \neq 0$, and we can define $f\colon Span\{x, y\} = Span\{x\} \to \mathbb{F}$ by $f(\alpha x) = \alpha$.

   If x and y are linearly independent, define $f\colon Span\{x, y\} \to \mathbb{F}$ by $f(\alpha x + \beta y) = \alpha + \beta$.

   Since $Span\{x, y\}$ is finite-dimensional in either of the above two cases, f is a bounded linear map. By the Hahn-Banach theorem, we have $F \in V^*$ such that $F|_{Span(x,y)} = f$ and $\|F\| = \|f\|$. Note that $f(x) \neq f(y)$, so $F(x) \neq F(y)$. The result now follows.

   (c) Let $x \in V$. Define $f\colon Span\{x\} \to \mathbb{F}$ by $f(\alpha x) = \alpha\|x\|$. Then $\|f\| = 1$, $|f(x)| = \|x\|$. By the Hahn-Banach theorem, we have $F \in V^*$ such that $F|_{Span(x)} = f$ and $\|F\| = \|f\| = 1$. In particular, $F(x) = \|x\|$.

   So by the above, if $|g(x)| \leq c$ whenever $g \in V^*$ with $\|g\| \leq 1$, then in particular $|F(x)| \leq c$. But $F(x) = \|x\|$, so $\|x\| \leq c$ are required.

2. Let P be the set of all subspaces $U \subseteq V$ such that $U \cap W = \{0\}$. Then P is a partially ordered set, with relation $\subseteq$.

   Let Q be a totally ordered subset of P. Let $U_0 = \cup_{U \in Q} U$. Certainly $0 \in U_0$. Let $x, y \in U_0$, $\alpha, \beta \in \mathbb{F}$.

   Then $x \in U_1$ and $y \in U_2$ for some $U_1, U_2 \in Q$. Since Q is totally ordered, either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$. Suppose $U_1 \subseteq U_2$.

   Then $x, y \in U_2$. Since U_2 is a subspace, we have that $\alpha x + \beta y \in U_2 \subseteq U_0$, so U_0 is a subspace of V. Since $U \cap W = \{0\}$ for all $U \in Q$, we have that $U_0 \cap W = \{0\}$. Hence $U_0 \in P$, and U_0 is an upper bound for Q.

   By Zorn's lemma, the set P therefore has a maximal element, Z. We claim that $Z + W = V$.

   Suppose $Z + W \neq V$. Then we have a vector $v \in V$ such that $v \notin Z + W$. Let $Z' = \{z + \alpha v \mid z \in Z, \alpha \in \mathbb{F}\}$.

   Then Z' is a subspace of V and $Z' \cap W = \{0\}$. Further, $Z \subseteq Z'$ and $Z \neq Z'$. This contradicts Z' being the maximal element of P.

Hence $Z + W = V$ and we are done.

3. By the formula

$$\cos(mx)\cos(nx) = \frac{1}{2}(\cos(m+n)x + \cos(m-x)x)$$

we see that A is closed under multiplication. The set A is certainly a vector space, so A is a subalgebra of $C_{\mathbb{R}}[0,\pi]$. It contains the constant functions.

Let $x, y \in [0,\pi]$ with $x \neq y$. Then $\cos x \neq \cos y$, so A separates points. Hence by the Stone-Weierstrass theorem, the set A is dense in $C_{\mathbb{R}}[0,\pi]$.

4. We can define a linear map $g \colon Span\{v_1, \ldots, v_n\} \to \mathbb{F}$ by the formula

$$g(\alpha_1 v_1 + \cdots + \alpha_n v_n) = \alpha_1 \lambda_1 + \cdots + \alpha_n \lambda_n$$

Observe that $g(v_i) = \lambda_i$ for each i. Since the space $Span\{v_1, \ldots, v_n\} \to \mathbb{F}$ is finite-dimensional, the map g is a bounded linear map.

Hence by the Hahn-Banach theorem, we can extend g to a bounded linear map $f \colon V \to \mathbb{F}$ such that $f(v_i) = \lambda_i$ for all i.