Functional Analysis: Problem Sheet 2 Solutions

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1. Let V and W be normed vector spaces.

   (a) Linearity is easy to check. Let $v, w \in V$. Then $\|(v,w)\| = \|v\| + \|w\|$. Hence
   $$\| + (v,w)\| = \|v+w\| \leq \|v\| + \|w\| = \|(v,w)\|.$$
   Hence the map $+$ is bounded.

   (b) Linearity is again clear.
   Fix $\lambda \in \mathbb{F}$. Let $v \in V$. Then
   $$\|M_\lambda(v)\| = \|\lambda v\| = |\lambda|\|v\|.$$
   Hence M_λ is bounded. The proof is similar for M_v.

   (c) Let $T\colon V \to W$ be a bounded linear map. Then there is a constant $M \geq 0$ such that $\|Tv\| \leq M\|v\|$ for all $v \in V$.
   Let $B \subseteq V$ be bounded. Then there is a constant $C \geq 0$ such that $\|v\| \leq C$ for all $v \in B$. Hence, let $w \in T[B]$. Write $w = T(v)$ where $v \in B$. Then $\|v\| \leq C$, so
   $$\|w\| = \|Tv\| \leq M\|v\| \leq MC.$$
   We conclude that $T[B]$ is bounded.
   Conversely, suppose $T[B]$ is bounded for any bounded subset $B \subseteq V$. Let $B = \{v \in V \mid \|v\| \leq 1\}$. Then $T[B]$ is bounded. Hence
   $$\sup\{\|Tv\| \mid v \in B\} = \sup\{\|Tv\| \mid \|v\| \leq 1\} < \infty$$
   which means that T is a bounded linear map.

2. Define linear maps $S, T\colon C[0,1] \to C[0,1]$ by the formulae
   $$S(f)(s) = \int_0^s f(t)\ dt \qquad T(g)(s) = sg(s).$$

   (a) Observe $|f(t)| \leq \|f\|$ for all $t \in [0,1]$. Hence
   $$\|S(f)\| = \sup\{\left|\int_0^s f(t)\ dt\right| \mid s \in [0,1]\} \leq \int_0^1 |f(t)|\ dt \leq \|f\|.$$

So S is a bounde d linear map, with $\|S\| \leq 1$. Let $f(t) = 1$ for all $t \in [0,1]$. Then $\|f\| = 1$, and

$$S(f)(s) = \int_0^s 1\ dt = s$$

and

$$\|S(f)\| = \sup\{s \mid s \in [0,1]\} = 1 = \|f\|.$$

So $\|S\| = 1$.

Now

$$\|T(f)\| = \sup\{|sf(s)| \mid s \in [0,1]\} \leq \sup \sup\{|f(s)| \mid s \in [0,1]\} = \|f\|.$$

If $f(t) = 1$ for all $t \in [0,1]$ then $T(f)(s) = s$, so as above $\|Tf\| = \|f\|$.
We conclude that T is a bounded linear map, and $\|T\| = 1$.

(b) Observe

$$ST(f) = S(T(f)) = \int_0^s tf(t)\,dt \qquad TS(f) = T(S(f)) = s\int_0^s f(t)\,dt.$$

These are not equal, so S and T do not commute.
Let $f \in C[0,1]$. Then

$$\|S(T(f))\| = \sup\left\{\left|\int_0^s tf(t)\ dt\right| \mid s \in [0,1]\right\} \leq \int_0^1 t|f(t)|\,dt \leq \|f\| \int_0^1 t\,dt = \frac{1}{2}\|f\|$$

so $\|ST\| \leq \frac{1}{2}$.
Let us try our earlier example, $f(t) = 1$ for all $t \in [0,1]$. Then

$$ST(f)(s) = \int_0^s t\ dt = \frac{1}{2}s^2 \qquad TS(f)(s) = s^2.$$

so

$$\|ST(f)\| = \frac{1}{2} \qquad \|f\| = \frac{1}{2}.$$

We conclude $\|ST\| = \frac{1}{2}$.
On the other hand

$$\|T(S(f))\| = \sup\left\{\left|s\int_0^s f(t)\ dt\right| \mid s \in [0,1]\right\} \leq \int_0^1 |f(t)|\ dt = \|f\|.$$

Try again $f(t) = 1$ for all $t \in [0,1]$. Then

$$TS(f)(s) = s\int_0^1 1\ dt = s^2$$

so $\|TS(f)\| = 1 = \|f\|$ and $\|TS\| = 1$.

3. Let P be the set of all polynomial functions.

(a) Certainly

$$\begin{aligned}\|\alpha(a_0 + a_1x + \cdots + a_nx^n)\| &= \max(|\alpha a_1|, \ldots, |\alpha a_n|) \\ &= |\alpha| \cdot \|a_0 + a_1x + \cdots + a_nx^n\|\end{aligned}$$

Let $a_0 + a_1x + \cdots + a_mx^m, b_0 + b_1x + \cdots + b_nx^n \in P$. By choosing some of the coefficients a_i or b_j to be zero, let us assume that $m = n$. Then

$$\begin{aligned}\|(a_0 + a_1x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_nx^n)\| &= \max(|a_1 + b_1|, \ldots, |a_n + b_n|) \\ &\leq \max(|a_1| + |b_1|, \ldots, |a_n| + |b_n|) \\ &\leq \max(|a_1|, \ldots, |a_n|) + \max(|b_1|, \ldots, |b \\ &\leq \|a_0 + a_1x + \cdots + a_nx^n\| + \|b_0 + b_1x\end{aligned}$$

Finally, let $\|a_0+a_1x+\cdots+a_nx^n\| = 0$. Then $\max(|a_1|, \ldots, |a_n|) = 0$, meaning $a_i = 0$ for all i, and so $a_0 + a_1x + \cdots + a_nx^n = 0$.
We see that the formula defines a norm.

(b) Let $A\colon P \to P$ and $B\colon P \to P$ be linear maps defined by the formulae

$$A(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = a_0 + a_1x + 2a_2x^2 + \cdots + na_nx^n$$

and

$$B(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = a_0 + a_1x + \frac{1}{2}a_2x^2 + \cdots + \frac{1}{n}a_nx^n$$

respectively.
Certainly

$$\begin{aligned}AB((a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) &= A(a_0 + a_1x + \tfrac{1}{2}a_2x^2 + \cdots + \tfrac{1}{n}a_nx^n) \\ &= a_0 + a_1x + \tfrac{1}{2}a_2x^2 + \cdots + \tfrac{1}{n}a_nx^n\end{aligned}$$

so AB is the identity map on P. Similarly, BA is the identity, so B is invertible, with inverse A.

(c) Observe

$$\begin{aligned}\|B(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)\| &= \max(|a_0|, |a_1|, \tfrac{|a_2|}{2}, \ldots \tfrac{|a_n|}{n}) \\ &\leq \max(|a_0|, \ldots, |a_n|) \\ &= \|a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\|\end{aligned}$$

Hence B is a bounded map. Observe that $\|x^n\| = 1$ and $\|A(x^n)\| = \|nx^n\| = n$.
Hence the supremum

$$\sup_{f \in P} \frac{\|Af\|}{\|f\|} \geq \frac{\|Ax^n\|}{\|x^n\|} = n$$

for all n, meaning the above supremum is not finite, and so A is not bounded.

(d) The open mapping theorem tells us that if P is a Banach space, and $B\colon P \to P$ is a bounded linear map, then $A = B^{-1}$ is also a bounded linear map.

But by the above, B is a bounded linear map, and A is not. Therefore P is not a Banach space, that is to say P is not complete.

4. Let V and W be normed vector spaces, and let $T\colon V \to W$ be a linear map.

(a) Suppose $\|Tv\| \geq m\|v\|$ for all $v \in V$. Let $U \subseteq V$ be an open set. Let $w \in T[U]$. Write $w = T(v)$, where $v \in U$. Then, since U is open, there exists $r > 0$ such that $B(v, r) \subseteq U$.

Let $r' = r/m$. Suppose $\|w' - w\| < r'$. Set $w' = T(v')$. Then $\|T(v' - v)\| < r'$. If follows that $m\|v - v'\| < r'$, and so $\|v - v'\| < r$. We conclude that $v' \in B(v, r)$, and so $T(v') \in T[U]$. Hence T is an open mapping.

Conversely, let T be open. Then $T[B(0, 1)]$ is open. Certainly $0 \in T[B(0, 1)]$, so there exists $m > 0$ such that if $\|w\| < m$, then $w \in T[B(0, 1)]$. Set $w = T(v)$. Then $\|v\| < 1$.

In other words, if $\|Tv\| < m$, then $\|v\| < 1$. Now, let $u \in V$. Define $u' = \frac{mu}{2\|Tu\|}$. Then $\|Tu'\| = \frac{m}{2} < m$, so $\|u'\| < 1$. In other words

$$\|Tu\| \geq \frac{m}{2}\|u\|$$

and we are done.

(b) Let T be open. Then $T[B(0, 1)]$ is open, and $0 \in T[B(0, 1)]$. So there exists $\delta > 0$ such that $B(0, \delta) \subseteq T[B(0, 1)]$.

Conversely, suppose there exists $\delta > 0$ such that$B_W(0, \delta) \subseteq T[B_V(0, 1)]$. Let $v \in V$, and let $v' = \frac{\delta v}{2\|Tv\|}$. Then $Tv' \in B_W(0, \delta)$, so $v' \in B_V(0, 1)$. Hence $\|v'\| < 1$, and

$$\|Tv\| \geq \frac{\delta}{2}\|v\|.$$

By the previous part of the question, T is open.

(c) The map B in the previous question is such an example.