Functional Analysis: Semester 2 Chapter 1 Solutions

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1. (a)

$$\mathfrak{F}\{e^{i\alpha x}f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} f(x) e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-i(\omega-\alpha)x}\, dx = \hat{f}(\omega-\alpha).$$

   (b)

$$\mathfrak{F}\{f(x-x_0)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-x_0)e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)e^{-i\omega(y+x_0)}\, dy = \hat{f}(\omega)e^{-i\omega x_0}$$

   (c)

$$\mathfrak{F}\{f(\alpha x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\alpha x)e^{-i\omega x}\, dx = \frac{1}{\alpha\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)e^{-i(\omega/\alpha)y}\, dy = \frac{1}{\alpha}\hat{f}\left(\frac{\omega}{\alpha}\right)$$

   (d) The result is immediate from linearity of integration.

2. Integrating by parts, we have

$$\int_a^b e^{-i\omega x} f'(x)\, dx + \left[e^{-i\omega x}f(x)\right]_a^b + i\omega\int_a^b e^{-i\omega x} f(x)\, dx.$$

Now, let $a \to -\infty$ and $b \to \infty$. Then the first term on the right converges to zero, as f is integrable and piecewise-continuous, so $f(x) \to 0$ as $x \to \pm\infty$. Hence, dividing by $\sqrt{2\pi}$, we see that the Fourier transform of f' exists, and

$$\mathfrak{F}\{f'(x)\}(\omega) = i\omega\hat{f}(\omega).$$

3. Let $f(x) = e^{-\alpha|x|}$. Then

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha|x|}e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-\alpha x}e^{-i\omega x}\, dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{\alpha x}e^{-i\omega x}\, dx.$$

Hence

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{(-\alpha-i\omega)x} + e^{(-\alpha+i\omega)x}\, dx = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\alpha+i\omega} + \frac{1}{\alpha-i\omega}\right) = \frac{2\alpha}{\sqrt{2\pi}(\alpha^2+\omega^2)}.$$

4. (a) Let $g(x) = \exp(-x^2)$. Then as in the notes,

$$\hat{g}(\omega) = \frac{1}{\sqrt{2}} \exp\left(-\frac{\omega^2}{4}\right).$$

Now

$$g_\sigma(x) = \frac{1}{\sigma} g\left(\frac{x}{\sqrt{2}\sigma}\right)$$

from which it follows, with question 2, that

$$\hat{g_\sigma}(\omega) = \sqrt{2}\sigma \hat{g}(\sqrt{2}\sigma\omega) = \exp\left(-\frac{\omega^2\sigma^2}{2}\right).$$

(b) By the convolution formula

$$\mathfrak{F}\{g_\sigma * g_\tau\}(\omega) = \hat{g_\sigma}(\omega)\hat{g_\tau}(\omega) = \exp\left(-\frac{\omega^2(\sigma^2+\tau^2)}{2}\right) = \hat{g}_{\sqrt{\sigma^2+\tau^2}}(\omega).$$

Applying the Fourier inversion theorem, we see that

$$g_\sigma * g_\tau = g_{\sigma^2+\tau^2}$$

as desired.

5. Without loss of generality, suppose $f(x) = g(x) = 0$ if $x \notin [a, b]$. Then

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-ixy} f(y)\ dy \qquad \hat{g}(x) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-ixy} g(y)\ dy$$

so

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)\ dx = \frac{1}{\sqrt{2\pi}} \int_a^b \int_a^b e^{-ixy} f(x)g(y)\ dx\ dy$$

and

$$\int_{-\infty}^{\infty} \hat{f}(x)g(x)\ dx = \frac{1}{\sqrt{2\pi}} \int_a^b \int_a^b e^{-ixy} f(y)g(x)\ dx\ dy.$$

Swapping the order of integration (which is allowed for piecewise-continuous functions on a compact domain such as $[a, b]\times[a, b]$), the result follows.

6. Applying the Fourier inversion theorem, we have

$$\phi(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega x}\ d\omega = \frac{1}{2\pi i x}(e^{i\pi x} - e^{-i\pi x}) = \frac{\sin(\pi x)}{\pi x}.$$

Hence

$$\phi_{0,k}(x) = \phi(x-k) = \frac{\sin(\pi(x-k))}{\pi(x-k)}.$$

Now, the Fourier transform is a unitary transformation. Hence

$$\|\phi_{0,k}\|^2 = \|\phi\|^2 = \|\hat{\phi}\|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi_{[-\pi,\pi]}(x)\ dx = 1.$$

Now $\phi_{0,k}(x) = \phi(x-k)$, so

$$\hat{\phi_{0,k}}(\omega) = \hat{\phi}(\omega)e^{-ik\omega} = \frac{1}{\sqrt{2\pi}}e^{-ik\omega}\chi_{[-\pi,\pi]}.$$

Therefore, for $k \neq l$

$$\langle \phi_{0,k}, \phi_{0,l}\rangle = \langle \hat{\phi_{0,k}}, \hat{\phi_{0,l}}\rangle = \int_{-\pi}^{\pi} e^{ik\omega}e^{-il\omega}\ d\omega = 0.$$

Thus the set $\{\phi_{0,k} \mid k \in \mathbb{Z}\}$ is orthonormal, as desired.

7. Let

$$H(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) We have

$$\hat{H}(\omega) = \frac{1}{\sqrt{2\pi}}\int_0^{\frac{1}{2}} e^{-i\omega x}\ dx - \frac{1}{\sqrt{2\pi}}\int_{\frac{1}{2}}^{1} e^{-i\omega x}\ dx$$

that is

$$\hat{H}(\omega) = -\frac{1}{\sqrt{2\pi}i\omega}\left[e^{-i\omega x}\right]_0^{\frac{1}{2}} + \frac{1}{\sqrt{2\pi}i\omega}\left[e^{-i\omega x}\right]_{\frac{1}{2}}^{1} = \frac{1}{\sqrt{2\pi}i\omega}(1-2e^{-i\omega/2}+e^{-i\omega}).$$

Now,

$$\begin{aligned} 1 - 2e^{-i\omega/2} + e^{i\omega} &= (1-e^{-i\omega/2})^2 = (e^{-i\omega/4})^2(e^{i\omega/4} - e^{-i\omega/4})^2 \\ &= e^{-i\omega/2}(2i\sin(\omega/4))^2 = -4e^{-i\omega/2}\sin^2(\omega/4). \end{aligned}$$

Therefore

$$\hat{H}(\omega) = \frac{-2e^{-i\omega/2}\sin^2(\omega/4}{)}\pi i\omega.$$

(b) Certainly, $H \in L^2(\mathbb{R})$. It remains to show that

$$\int_{-\infty}^{\infty} \frac{|\hat{H}(\omega)|^2}{|\omega|}\ d\omega < \infty.$$

Let $f(\omega) = \frac{\hat{H}(\omega)^2}{\omega}$. Then

$$|f(\omega)| = \frac{4\sin^4(\omega/4)}{\pi^2\omega^3}.$$

Therefore $|f(\omega)| \to 0$ as $\omega \to 0$, so f is a bounded continuous function, and the integral

$$\int_{-1}^{1} f(\omega)\ d\omega$$

exists.

Observe

$$f(\omega) \leq \frac{C}{|\omega|^3}$$

where C is constant, so

$$\int_{1}^{\infty} f(\omega)\ d\omega < \infty \qquad \int_{-\infty}^{-1} f(\omega)\ d\omega < \infty.$$

Putting all of this together, we see

$$\int_{-\infty}^{\infty} f(\omega)\ d\omega < \infty$$

and we are done.

8. We first show that $\psi * \phi$ is a well-defined L^2-integrable function. We want to show that the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi * \phi(x)| = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} \psi(x-t)\phi(t)\ dt\right|\ dx$$

is finite.

Thus

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi * \phi(x)| \leq \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi(x-s)\phi(s)\psi(x-t)\phi(t)|\ ds\ dt\ dx$$

Swapping integral signs is allowed here, so the integral is equal to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi(x-s)\phi(s)\psi(x-t)\phi(t)|\ dx\ ds\ dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\phi(s)\phi(t)| \int_{-\infty}^{\infty} |\psi(x-s)\psi(x-t)|\ dx\ ds\ dt.$$

We know that $\psi \in L^2(\mathbb{R})$, so we have L^2-integrable functions $x \mapsto \overline{\psi}(x-s)$ and $x \mapsto \psi(x-t)$ for each $s, t \in \mathbb{R}$. Hence the inner product

$$\langle \overline{\psi(x-s)}, \psi(x-t) \rangle = \int_{-\infty}^{\infty} \psi(x-s)\psi(x-t)\ dt$$

is well-defined.

Observe that the functions $\overline{\psi}(x-s)$ and $\psi(x-t)$ both have norm $\|\psi\|$. Hence, by the Cauchy-Schwarz inequality

$$\left|\int_{-\infty}^{\infty} \psi(x-s)\psi(x-t)\ dt\right| \leq \|\psi\|^2$$

so

$$\int_{-\infty}^{\infty} |\psi * \phi(x)|^2\ dx \leq \frac{\|\psi\|^2}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi(s)\phi(t)|\ ds\ dt = \frac{\|\psi\|^2\|\phi\|^2}{2\pi}$$

as $\phi \in L^1(\mathbb{R})$.

Hence $\psi * \phi$ is a well-defined L^2-integrable function.

We now need to check

$$\int_{-\infty}^{\infty} \frac{|\psi \hat{*} \phi(\omega)|^2}{|\omega|}\ d\omega < \infty.$$

By the convolution formula for the Fourier transform, $\psi \hat{*} \phi(\omega) = \hat{\psi}(\omega)\hat{\phi}(\omega)$, so

$$\int_{-\infty}^{\infty} \frac{|\psi \hat{*} \phi(\omega)|^2}{|\omega|}\ d\omega = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}|\hat{\phi}(\omega)|^2\ d\omega.$$

Now the Fourier transform of a piecewise-continuous function is bounded, so we have a constant M such that $|\hat{\phi}(\omega)|^2 \leq M$ for all $\omega \in \mathbb{R}$. Hence

$$\int_{-\infty}^{\infty} \frac{|\psi \hat{*} \phi(\omega)|^2}{|\omega|}\ d\omega \leq M\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\ d\omega < \infty.$$

Thus $\psi * \phi$ is an admissible wavelet.

9. Let $s > 0$. Then

$$\psi_{s,t}(x) = \frac{1}{\sqrt{s}}\psi\left(\frac{x-t}{s}\right).$$

Using properties of the Fourier transform

$$\hat{\psi_{s,t}}(\omega) = \frac{1}{\sqrt{s}}s\mathfrak{F}\{\psi(x-t)\}(s\omega) = \sqrt{s}\mathfrak{F}\{\psi(x-t)\}(s\omega) = \sqrt{s}e^{-it\omega}\hat{\psi}(s\omega).$$

Let $s < 0$. Set $s' = -s$. Then

$$\psi_{s,t}(x) = \frac{1}{\sqrt{s'}}\psi\left(-\frac{x-t}{s}\right).$$

Using properties of the Fourier transform

$$\hat{\psi_{s,t}}(\omega) = \frac{1}{\sqrt{s'}}s'\mathfrak{F}\{\psi(-(x-t))\}(s'\omega) = \sqrt{s'}\mathfrak{F}\{\psi(x-t)\}(-s'\omega) = \sqrt{s'}e^{-it\omega}\hat{\psi}(s\omega).$$

In either case

$$\hat{\psi}_{s,t}(\omega) = \sqrt{|s|}e^{-it\omega}\hat{\psi}(s\omega)$$

as required.