Functional Analysis: Problem Sheet 1- Solutions

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1. (a) Let $x, y \in X$. By the triangle inequality, we have

$$\|x\| = \|(x-y)+y\| \leq \|x-y\| + \|y\|,$$

so

$$\|x\| - \|y\| \leq \|x-y\|.$$

Similarly

$$\|y\| - \|x\| \leq \|y-x\| = \|x-y\|$$

so $|\|x\| - \|y\|| \leq \|x-y\|$ and we are done.

(b) Let $x \in X$ and $\varepsilon > 0$. Set $\delta = \varepsilon$. Suppose $y \in X$, and $\|x-y\| < \delta$. Then by the above, $|\|x\| - \|y\|| < \varepsilon$. It follows that the norm is a continuous map.

(c) Let (x_n, y_n) be a sequence in $V \oplus V$ which converges in norm to (x, y). Then as in question 2, we know that $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$ as $n \to \infty$.

Hence by the triangle inequality

$$\|(x_n + y_n) - (x+y)\| \leq \|x_n - x\| + \|y_n - y\| \to 0$$

as $n \to \infty$.

It follows that $(x_n + y_n)$ converges in norm to $x + y$, which means addition is continuous.

We can similarly prove that scale multiplication is a continuous operation.

2. Let X and Y be normed vector spaces.

(a) Let $x \in X$, $y \in Y$, and $\alpha \in \mathbb{F}$. Then

$$\|\alpha(x,y)\| = \|\alpha x\| + \|\alpha y\| = |\alpha| \cdot \|x\| + |\alpha| \cdot \|y\| = |\alpha| \cdot \|(x,y)\|.$$

Let $u, x \in X$ and $v, y \in Y$. Observe

$$\|(u,v)+(x,y)\| = \|u+x\|+\|v+y\| \leq \|u\|+\|x\|+\|v\|+\|y\| = \|(u,v)\|+\|(x,y)\|.$$

Finally, suppose $\|(x,y)\| = 0$. Then $\|x\| + \|y\| = 0$. But $\|x\| \geq 0$ and $\|y\| \geq 0$. Thus $\|x\| = \|y\| = 0$, which means $x = 0$ and $y = 0$, that is $(x,y) = (0,0)$.

(b) Let $\varepsilon > 0$. Then we have $N_X, N_Y \in \mathbb{N}$ such that $\|x_n - x\| < \frac{\varepsilon}{2}$ whenever $n \geq N_X$, and $\|y_n - y\| < \frac{\varepsilon}{2}$ whenever $n \geq N_Y$. Let $N = \max(N_X, N_Y)$. Let $n \geq N$. Then

$$\|(x_n, y_n) - (x, y)\| = \|x_n - x\| + \|y_n - y\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So the sequence (x_n, y_n) converges in in to (x, y) as desired.

(c) Let (x_n, y_n) be a Cauchy sequence in $X \times Y$. Observe that for all $m, n \in \mathbb{N}$, we have

$$\|(x_m, y_m) - (x_n, y_n)\| = \|x_m - x_n\| + \|y_m - y_n\|$$

so

$$\|x_m - x_n\| \leq \|(x_m, y_m) - (x_n, y_n)\| \qquad \|y_m - y_n\| \leq \|(x_m, y_m) - (x_n, y_n)\|.$$

It follows that the sequences (x_n) and (y_n) are also Cauchy. Since the spaces X and Y are complete, these sequences have limits, x and y respectively.

By the previous part of the question, the sequence (x_n, y_n) therefore has limit (x, y). It follows that the space $X \times Y$ is a Banach space.

(d) Let $\mathbb{F}$ denote $\mathbb{R}$ or $\mathbb{C}$ as usual. Then $\mathbb{F}$ is a Banach space.

If $\mathbb{F}^n$ is a Banach space, then by the above so is $\mathbb{F}^{n+1} = \mathbb{F}^n \oplus \mathbb{F}$. It follows that for all $n \in \mathbb{N}$, $\mathbb{F}^n$ is a Banach space.

3. (a) Let $(a_n) \in c_0$. Since $a_n \to 0$ as $n \to \infty$, the sequence (a_n) os bounded, and

$$\|(a_n)\| = \sup\{|a_n| \mid n \in \mathbb{N}\}$$

is well-defined.

To check this formular defines a norm:

- Let $\alpha \in \mathbb{F}$ and $(a_n) \in c_0$. Then

$$\|\alpha(a_n)\| = \sup\{|\alpha a_n| \; n \in \mathbb{N}\} = |\alpha| \sup\{|a_n| \mid n \in \mathbb{N}\} = |\alpha| \cdot \|(a_n)\|.$$

- Let $(a_n), (b_n) \in c_0$. Then

$$\begin{aligned} \|(a_n) + (b_n)\| &= \sup\{|a_n + b_n| \mid n \in \mathbb{N}\} \\ &\leq \sup\{|a_m + b_n| \mid m, n \in \mathbb{N}\} \\ &\leq \sup\{|a_m| + |b_n| \mid m, n \in \mathbb{N}\} \\ &= \sup\{|a_m| \mid m \in \mathbb{N}\} + \sup\{|b_n| \mid n \in \mathbb{N}\} \\ &= \|(a_n)\| + \|b_n\| \end{aligned}$$

- Suppose $\|(a_n)\| = 0$. Then $\sup\{|a_n| \mid n \in \mathbb{N}\} = 0$.
  Since $|a_n| \geq 0$ for all n, this means $|a_n| = 0$ for all n, and so $(a_n) = 0$.

So we have a norm.

(b) Let (X_k) be a Cauchy sequence in c_0. Write $X_k = (a_n^k)_{n=1}^\infty$.
Let $\varepsilon > 0$. Then there exists K such that if $k, l \geq K$ then $\|X_k - X_l\| < \varepsilon$.
But this means that

$$\sup\{|a_n^k - a_n^l| \mid n \in \mathbb{N}\} < \varepsilon$$

for $k, l \geq K$. In particular, for each n, $|a_n^k - a_n^l| < \varepsilon$ whenever $k, l \geq K$.
Hence, for each n, $(a_n^k)_{k=1}^\infty$ is a Cauchy sequence (in k). Since $a_n^k \in \mathbb{C}$, and $\mathbb{C}$ is complete, the sequence $(a_n^k)_{k=1}^\infty$ converges to some limit b_n.
Let $X = (b_n)$. We claim that $\|X_k - X\| \to 0$ as $k \to \infty$, and $X \in c_0$. Hence c_0 is a Banach space.
For the first, let $\varepsilon > 0$. Then we have $K \in \mathbb{N}$ such that for each n, $|a_n^k - a_n^l| < \frac{\varepsilon}{2}$ whenever $k, l \geq K$. If we let $l \to \infty$, we see that for each n, we have $|a_n^k - b_n| \leq \frac{\varepsilon}{2}$ whenever $k \geq K$.
It follows that $\|X_k - X\| \leq \frac{\varepsilon}{2} < \varepsilon$ whenever $k \geq K$. This means that $\|X_k - X\| \to 0$ as $k \to \infty$.
It remains to check that $X \in c_0$. Again, let $\varepsilon > 0$. Pick K such that $\|X_k - X\| < \frac{\varepsilon}{2}$ for $k \geq K$. For such k, write $X_k = (a_n^k)_{n=1}^\infty$. Then since $X_k \in c_0$, we have $n \in \mathbb{N}$ such that $|a_n^k| < \frac{\varepsilon}{2}$ whenever $n \geq N$.
Hence, for $n \geq N$, we have

$$|b_n| \leq |b_n - a_n^k| + |a_n^k| \leq \|X - X_k| + |a_n^k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We conclude that $X = (b_n) \in c_0$.

4. (a) The space c_0 is a Banach space, and a subspace of l^∞ with the same norm. Therefore c_0 is closed.

(b) Consider the sequences X_n where

$$X_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots)$$

Let

$$X = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$$

Then $X_n \in c_{00}$ and $\|X_n - X\| = \frac{1}{n+1} \to 0$ as $n \to \infty$. But $X \notin c_{00}$. So c_{00} is not closed.

(c) Let (X_k) be a sequence in C_N which converges to a limit $X \in l^\infty$. Write

$$X = (a_1, \ldots, a_N, a_{N+1}, a_{N+2}, \ldots).$$

Suppose $a_n \neq 0$ for some $n > N$. Then, since $X_k \in c_N$, we have $\|X_k - X\| \geq |a_n|$ for all k. It follows that X cannot be the limit of the sequence (X_k).
It follows that $X \in c_N$, and c_N is closed.