Functional Analysis: Problem Booklet

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Chapter 1

1. Suppose that X is a vector space equipped with a translation-invariant and dilation-invariant metric, d. Prove that there is a norm defined by the formula
$$\|x\| = d(x, 0)$$
which induces the metric d.

2. Let X be a normed vector space.

   (a) Prove that
   $$\|x - y\| \geq |\|x\| - \|y\||$$
   for all vectors $x, y \in X$.

   (b) Prove that the norm defines a continuous map $X \to \mathbb{R}^{\geq 0}$ by writing $x \mapsto \|x\|$.

3. Let X and Y be normed vector spaces.

   (a) Show that we can define a norm on the product $X \times Y$ by the formula
   $$\|(x, y)\| = \|x\| + \|y\| \qquad x \in X,\ y \in Y.$$

   (b) Let (x_n) and (y_n) be sequences in the spaces X and Y with norm limits x and y respectively. Show that the sequence (x_n, y_n) in $X \times Y$ converges in norm in $X \times Y$ to the limit (x, y).

   (c) Suppose that X and Y are Banach spaces. Is $X \times Y$ necessarily a Banach space? Justify your answer.

4. (a) Prove that we have a norm on the space $\mathbb{F}^k$ defined by the formula
   $$\|(\alpha^{(1)}, \ldots, \alpha^{(k)})\| = \max(|\alpha^{(1)}|, \ldots, |\alpha^{(k)}|)$$
   $\alpha^{(i)} \in \mathbb{F}$.

   (b) Is the space $\mathbb{F}^k$ equipped with this norm a Banach space ? Justify your answer.

5. Let P be the space of all real polynomials.

   (a) Prove that P is a normed vector space when equipped with the norm

   $$\|p\| = \sup\{|p(t)| \mid t \in [0,1]\} \qquad p \in P.$$

   (b) Is the space P a Banach space when equipped with the above norm ? Justify your answer.

6. Let X be a normed vector space.

   (a) Let $A \subseteq X$, $x \in X$. Prove that $x \in \overline{A}$ if and only if $B(x,\varepsilon) \cap A \neq \emptyset$ for all $\varepsilon > 0$.

   (b) Let $x \in X$ and $\delta > 0$. Show that

   $$\overline{B(x,\delta)} = \{y \in X \mid \|y - x\| \leq \delta\}.$$

7. Let $p > 1$. Define l^p to be the space of sequences, (a_k), in the field $\mathbb{F}$ such that $\sum_{k=1}^{\infty} |a_k|^p < \infty$. For a sequence $(a_k) \in l^p$, write

   $$\|(a_k)\|_p = \left(\sum_{k=1}^{\infty} |a_k|^p\right)^{\frac{1}{p}}.$$

   (a) Choose q such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $(a_k) \in l^p$ and $(b_k) \in l^q$. Show that

   $$\sum_{k=1}^{\infty} |a_k b_k| \leq \|(a_k)\|_p \|(b_k)\|_q.$$

   (b) Prove that the function $\| - \|_p$ defines a norm on the space l^p.

8. Let X be a normed vector space.

   (a) Let (x_n) be a Cauchy sequence in X with a convergent subsequence. Show that (x_n) converges.

   (b) We say a series $\sum_{n=1}^{\infty} x_n$ *converges* in X if the sequence of partial sums, (s_N), defined by writing $s_N = \sum_{n=1}^{N} x_n$, converges in norm.
   We say the series $\sum_{n=1}^{\infty} x_n$ *converges absolutely* if the series $\sum_{n=1}^{\infty} \|x_n\|$ converges in $\mathbb{R}$.
   Prove that the space X is a Banach space if and only if every series that converges absolutely also converges.
   [Hint: To prove X is a Banach space from the series property, consider a Cauchy sequence (x_n). Choose N_k such that $\|x_m - x_n\| < \frac{1}{2^k}$ whenever $m, n \geq N_k$, and consider the series $\sum_{n=1}^{\infty} y_n$ where $y_1 = x_{N_1}$, $y_2 = x_{N_2} - x_{N_1}$, $y_3 = x_{N_3} - x_{N_2}$, and so on.]

9. Use the previous question to show that the following spaces are Banach spaces. You may assume without proof that the stated formulae define norms.

   (a) The space l^1, consisting of all sequences (a_k) in the field $\mathbb{F}$ such that $\sum_{k=1}^{\infty} |a_k| < \infty$, with norm $\|(a_k)\| = \sum_{k=1}^{\infty} |a_k|$.

   (b) The space l^∞, consisting of all bounded sequences in the field $\mathbb{F}$, with norm $\|(a_k)\| = \sup\{|a_k| \mid k \in \mathbb{N}\}$.

10. (a) Let X be a Banach space. Prove that a linear subspace of X is closed if and only if it is a Banach space.

    (b) Let c_0 be the set of all sequences (a_k) in $\mathbb{F}$ such that $a_k \to 0$ as $k \to \infty$. We can define a norm on c_0 by the formula

    $$\|(a_k)\| = \sup\{|a_k| \mid k \in \mathbb{N}\}.$$

    Prove that c_0 is a Banach space.

11. (a) Let V be a finite-dimensional normed vector space. Let $\{e_1, \ldots, e_k\}$ be a basis for V.
    Show that there exists $c > 0$ such that

    $$\|\alpha_1 e_1 + \cdots + \alpha_k e_k\| \geq c(|\alpha_1| + \cdots + |\alpha_k|)$$

    for all scalars α_i.

    (b) Show that any finite-dimensional normed vector space is a Banach space.

12. Let l^2 be the set of all sequences (a_k) in the field $\mathbb{F}$ such that $\sum_{k=1}^{\infty} |a_k|^2 < \infty$, with norm $\|(a_k)\| = \left(\sum_{k=1}^{\infty} |a_k|^2\right)^{\frac{1}{2}}$.

    (a) Show that l^1 is a linear subspace of l^2.

    (b) Is l^1 a closed subspace of l^2 ? Justify your answer.

Chapter 2

1. Let X be a normed vector space.

   (a) Prove that the closure of a convex subset of X is convex.

   (b) Let $A, B \subseteq X$. We define the *sum*

   $$A + B = \{a + b \mid a \in A, b \in B\}.$$

   Is it true that if A and B are convex, then the sum $A + B$ has to be convex ? If so prove it; if not, give a counterexample.

(c) Let Y be another normed vector space, and let $T: X \to Y$ be a bounded linear map. Let $A \subseteq X$ be convex. Prove that the image $T[A]$ is convex.

Is this statement still true if the map T is not bounded ?

2. Let X and Y be normed vector spaces, and let $f: X \to Y$ be a map (not necessarily linear).

   (a) Prove that the map f is continuous if and only if whenever (x_n) is a convergent sequence in X, the sequence $(f(x_n))$ converges in Y, and

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right).$$

   (b) Let $K \subseteq X$ be compact, and let $f: X \to Y$ be continuous. Prove that the image $f[K]$ is compact.

3. Let X be a normed vector space.

   (a) Prove that the operation of addition defines a continuous map $+: X \times X \to X$.

   (b) Prove that the operation of scalar multiplication defines a continuous map $\times: F \times X \to X$.

4. Let X and Y be normed vector spaces. Prove that a linear map $T: X \to Y$ is bounded if and only if for any bounded subset $B \subseteq X$, the image $T[B]$ is bounded.

5. Let $C^\infty[0,1]$ be the space of infinitely differentiable functions on the interval $[0,1]$ equipped with the norm

$$\|f\| = \sup\{|f(x)| \mid x \in [0,1]\}.$$

   (a) Define a map $D: C^\infty[0,1] \to C^\infty[0,1]$ by differentiating a function. Is D bounded ? If not, prove it. If so, find its norm.

   (b) Define a map $I: C^\infty[0,1] \to C^\infty[0,1]$ by the formula $I(f)(x) = \int_0^x f(t)\, dt$. Is I bounded ? If not, prove it. If so, find its norm.

6. Let X be a normed vector space.

   (a) Prove that any compact subset of X is closed and bounded.

   (b) Is the converse of the above statement true ? If so, prove it; if not, give a counterexample.

7. Let V and W be normed vector spaces, and let $T: V \to W$ be a linear map. Prove that T is open if and only if there exists $\delta > 0$ such that

$$B_W(0,\delta) \subseteq T[B_V(0,1)] \subseteq W.$$

8. Give an example of a bounded and bijective linear map between normed vector spaces which is not open.

9. Show that an open mapping need not map closed sets to closed sets.

10. (a) Let $f\colon X \to Y$ be a bijective map between metric spaces. Show that the map f has closed graph if and only if the map f^{-1} has a closed graph.

    (b) Give an example of a map between metric spaces where the graph is not closed.

    (c) Is it true that any map between metric spaces which has a closed graph is continuous ? If so, prove it; if not, give a counterexample.

Chapter 3

1. Let X be a vector space, and let $\| - \|_1$ and $\| - \|_2$ be norms on X. We call these norms *equivalent* if there are constants $a, b > 0$ such that

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$$

for all $x \in X$.

    (a) Show that the notion of norms being equivalent is an equivalence relation.

    (b) Let $\| - \|_1$ and $\| - \|_2$ be equivalent norms on a vector space X. Let (x_n) be a sequence in X, and let $x \in X$. Show that (x_n) converges to x with respect to the norm $\| - \|_1$ if and only if (x_n) converges to x with respect to the norm $\| - \|_2$.

    (c) Let $\| - \|_1$ and $\| - \|_2$ be norms on a vector space X. Suppose that for every sequence (x_n) in X, convergence of (x_n) to a point $x \in X$ holds with respect to the norm $\| - \|_1$ if and only if it holds with respect to the norm $\| - \|_2$.

    Is it then true that the norms $\| - \|_1$ and $\| - \|_2$ are equivalent. If so, prove it; if not, give a counterexample.

2. Let X be a vector space, and let $\| - \|_1$ and $\| - \|_2$ be norms on X such that X is complete with respect to both of these norms.

    (a) Suppose we have a constant $c > 0$ such that $\|x\|_1 \leq c\|x\|_2$ for all $x \in X$. Show that the norms $\| - \|_1$ and $\| - \|_2$ are equivalent.

    (b) Show that any two norms on a finite-dimensional vector space are equivalent.

3. Let X be a normed vector space. Prove using Zorn's lemma that there is a linearly independent set S such that every vector $x \in X$ is a norm limit of vectors of the form
$$\alpha_1 x_1 + \cdots + \alpha_n x_n$$
where $\alpha_i \in \mathbb{F}$ and $x_i \in S$.

4. Let V be a vector space over the field $\mathbb{F}$, and let U be a subspace of V. Let $f\colon U \to \mathbb{F}$ be a linear map. Prove that there is a linear map $F\colon V \to \mathbb{F}$ such that $F|_U = f$.

5. (a) Let X be a normed vector space, and let $x \in X \backslash \{0\}$. Show that there is a linear functional $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.
   (b) Let X be a normed vector space, and $x \in X$. Show that
$$\|x\| = \sup_{f \in X^* \backslash \{0\}} \frac{|f(x)|}{\|x\|}.$$
   (c) Deduce that we have an isometry $\theta\colon X \to (X^*)^*$ defined by the formula
$$\theta(x)(f) = f(x) \qquad f \in X^*,\ x \in X.$$
   (d) Use this to prove that any normed vector space has a completion.

6. (a) Let $C^P[0, 2\pi]$ be the set of all continuous functions $f\colon [0, 2\pi] \to \mathbb{C}$ such that $f(0) = f(2\pi)$.
   Let $A \subseteq C^P[0, 2\pi]$ be the set of all functions of the form of the form
$$f(x) = \sum_{k=m}^{n} a_k e^{ikx} \qquad m, n \in \mathbb{Z},\ a_k \in \mathbb{C}.$$
   Prove that A is a dense subset of the space $C^P[0, 2\pi]$.
   (b) Let $f \in C^P[0, 2\pi]$. Let $\varepsilon > 0$. Prove that we have $N \in \mathbb{N}$ and $a_n, b_n \in \mathbb{R}$ such that
$$\left| \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)) - f(x) \right| < \varepsilon$$
   for all $x \in [0, 2\pi]$.

Chapter 4

1. Prove that we have an inner product on the space $C[0, 1]$ defined by the formula
$$\langle f, g \rangle = \int_0^1 \overline{f(t)} g(t)\ dt.$$

2. Let H be a Hilbert space, and let S be a subset of H.
   (a) Prove that $S^\perp$ is a linear subspace of H.
   (b) If $x \in S$ and $v \in S^\perp$, show that $\|x+v\|^2 = \|x\|^2 + \|v\|^2$.
   (c) Let $\{S_i \mid i \in I\}$ be a collection of subsets of H. Prove that
   $$\left(\bigcup_{i\in I} S_i\right)^\perp = \bigcap_{i\in I} S_i^\perp.$$
   (d) Let $S' \subseteq S$. Prove that $S^\perp \subseteq (S')^\perp$.
   (e) Let V be a subspace of H. Prove that $(\overline{V})^\perp = V^\perp$.

3. (a) Prove that the norm on an inner product space satisfies the parallelogram law.
   (b) Let V be a real inner product space. Prove that
   $$\langle u, v\rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right).$$
   (c) Let V be a real normed vector space, where the norm satisfies the parallelogram law. Define
   $$\langle u, v\rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right).$$
   Prove that
   $$\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$$
   for all $u, v, w \in V$.
   [Hint: First prove that $\langle u, v+w\rangle = 2\langle \frac{u}{2}, v\rangle + 2\langle \frac{u}{2}, w\rangle$.]
   (d) If $\langle -, -\rangle$ is defined as above, prove that
   $$\langle u, \alpha v\rangle = \alpha\langle u, v\rangle$$
   for all $u, v \in V$ and $\alpha \in \mathbb{R}$.
   [Hint: First prove the formula for $\alpha \in \mathbb{N}$. Then extend to $\mathbb{Q}$ algebraically, and use continuity of an appropriate function to extend to $\mathbb{R}$.]
   (e) Let V be a real normed vector space, where the norm satisfies the parallelogram law. Prove that the formula
   $$\langle u, v\rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right)$$
   defines an inner product on V.

   Note: Something similar to the above also works for complex normed vector spaces where the norm satisfies the parallelogram law, but the formula for the inner product is given by the more complicated formula
   $$\langle u, v\rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right) + \frac{i}{4}\left(\|u+iv\|^2 - \|u-iv\|^2\right).$$

4. Let H be a Hilbert space, let $x \in H$, and let $C \subseteq H$ be a closed convex subset. Suppose we have points $y, y' \in C$ such that $\|x - y\| = \|x - y'\| = d(x, C)$. Prove that $y = y'$.

   [Hint: Look at $\frac{1}{2}(y + y')$ and use the parallelogram law.]

5. (a) Let H be a Hilbert space. Prove that we have a conjugate-linear isometry $J: H \to H^*$ defined by writing

$$J(v)(x) = \langle v, x \rangle \qquad v \in H,\ x \in H.$$

   (b) Prove that the map J is invertible.

   (c) Let H be a Hilbert space. Prove that the spaces H and $(H^*)^*$ are isometrically isomorphic.

6. Let $S, T: H \to H$ be bounded linear maps, and let $\alpha, \beta \in \mathbb{F}$. Prove the following statements:

   - $(\alpha S + \beta T)^* = \overline{\alpha} S^* + \overline{\beta} T^*$.
   - $(S^*)^* = S$.
   - $(ST)^* = T^* S^*$.

7. (a) Let H be a Hilbert space, and let $y \in H$. Show that

$$\|y\| = \sup\{|\langle x, y \rangle| \mid \|x\| \leq 1\}.$$

   (b) Let $T: H \to H$ be a bounded linear map. Prove that the adjoint $T^*: H \to H$ is a bounded linear map, with norm $\|T^*\| = \|T\|$.

8. Let H be a Hilbert space, and let $P: H \to H$ be a bounded linear map, with closed image, V. Prove that the following are equivalent:

   - $P \circ P = P$, and $P^* = P$.
   - The map P is the orthogonal projection onto V.

9. Let $f \in C[0, 1]$. Prove that we can define a bounded linear map $M_f: L^2[0, 1] \to L^2[0, 1]$ by the formula

$$M_f(g)(x) = f(x)g(x)$$

   where $g \in C[0, 1]$.

   Show that if f is real-valued, then $M_f^* = M_f$.

10. Define $T: l^2 \to l^2$ by the formula

$$T(a_1, a_2, a_3, \ldots) = (0, a_1, 0, a_2, \ldots).$$

   Find T^*.

11. Let H be a Hilbert space. Let $f, g\colon H \to H$ be functions where
$$\langle f(v), w\rangle = \langle v, g(w)\rangle$$
for all $v, w \in H$. Prove that f and g are bounded linear maps.
[Hint: Use the closed graph theorem]

Chapter 5

1. (a) What does it mean to say an infinite set is linearly independent ?
   (b) Prove that any orthogonal set is linearly independent.
2. Prove that every Hilbert space has a basis.
   [Hint: You need to use Zorn's lemma]
3. (a) Let S be a subset of a Hilbert space H. Prove that $S^\perp = Span(S)^\perp$.
   (b) Let H be a Hilbert space, and let $(e_n)_{n=1}^\infty$ be an orthonormal set such that for any element $x \in H$, we have
$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle e_n, x\rangle|^2.$$
   Prove that (e_n) is an orthonormal basis of H.
4. Prove that the set of functions $\{\frac{1}{\sqrt{2\pi}} e^{ikx}\ k \in \mathbb{Z}\}$ is an orthonormal basis for the complex Hilbert space $L^2[0, 2\pi]$.
5. (a) Calculate the Legendre polynomials $P_0(t)$, $P_1(t)$, $P_2(t)$, $P_3(t)$, $P_4(t)$ and $P_5(t)$.
   (b) Prove that we can write the n-th Legendre polynomial
$$P_n(t) = \sum_{k=0}^{N} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} t^{n-2k}$$
   where $N = n/2$ when n is even, and $N = (n-1)/2$ when n is odd.
6. (a) Let
$$v_0(x) = \frac{1}{\sqrt{\pi}},\ v_1(x) = \sqrt{\frac{2}{\pi}} \cos x,\ v_2(x) = \sqrt{\frac{2}{\pi}} \cos(2x),\ v_3(x) = \sqrt{\frac{2}{\pi}} \cos(3x), \ldots$$
   Show that $\{v_0, v_1, v_2, v_3, \ldots\}$ is an orthonormal basis for $L^2[0, \pi]$.
   (b) Let $f(x) = x$. Find coefficients $a_k \in \mathbb{C}$ such that the series
$$\sum_{k=0}^{\infty} a_k v_k = f$$
   in the space $L^2[0, \pi]$.

(c) Use the above and Parseval's law to evaluate the series

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}$$

Chapter 6

1. Let $k \in \mathbb{Z}$. Define a path $\gamma_k \colon [0,1] \to \mathbb{C}\backslash\{0\}$ by the formula

$$\gamma_k(t) = \exp(2\pi i k t).$$

   (a) Let $U \subseteq \mathbb{C}$ be an open set containing the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$. Let $f \colon U \to \mathbb{C}$ be holomorphic. Show that

$$\oint_{\gamma_k} f(z)\, dz = k \oint_{\gamma_1} f(z)\, dz.$$

   (b) Calculate $Wind(\gamma_k, a)$ in each of the two cases $|a| < 1$ and $|a| > 1$.

2. Write down Cauchy's formula for derivatives.

   [Note: This formula is *not* in the notes; the exercise is to find out what it is.]

3. (a) Let $0 < r < R$, and let $f \colon B(0,R) \to \mathbb{C}$ be a holomorphic function. Let $M = \sup\{|f(z)| \mid |z| = r\}$. Use Cauchy's formula for derivatives to show that

$$|f'(w)| \leq \frac{Mr}{(r - |w|)^2}$$

   for all $w \in B(0,r)$.

   (b) Use the above to prove Liouville's theorem.

4. (a) Let $p \colon \mathbb{C} \to \mathbb{C}$ be a non-constant polynomial (with complex coefficients). Prove that $p(z) \to \infty$ as $z \to \infty$.

   (b) Let $f \colon \mathbb{C} \to \mathbb{C}$ be a function. Prove that $\lim_{z\to\infty} f(z) = \infty$ if and only if $\lim_{z\to\infty} \frac{1}{f(z)} = 0$.

   (c) Use the above, and Liouville's theorem to prove the *fundamental theorem of algebra*, which states that for every non-constant complex polynomial $p \colon \mathbb{C} \to \mathbb{C}$, there is at least one point $z_0 \in \mathbb{C}$ for which $p(z_0) = 0$.

5. Let V be a complex Banach space, let $D \subseteq V$ be an open set.

   (a) Let $f \colon D \to V$ be a holomorphic function. Prove that f is continuous.

   (b) Let $g \colon D \to V$ be a holomorphic function such that $g'(z) = 0$ for all $z \in D$. Prove that g is constant.

Chapter 7

1. Let V be a Banach space. Let $S, T: V \to V$ be bounded linear maps. Prove that $\|ST\| \leq \|S\| \cdot \|T\|$. Deduce that the set of bounded linear operators on V, $\mathcal{L}(V)$, is a Banach algebra.

2. Let A be a unital Banach algebra.

   (a) Let $x \in A$, and $\lambda \in \mathbb{F}$. Suppose that $\|x\| < |\lambda|$. Show that the element $\lambda - x$ is invertible.

   (b) Show that the set of invertible elements in A is open.

3. Let A be a unital Banach algebra. Let $x \in A$. Show that $\|R_\lambda(x)\| \to 0$ as $\lambda \to \infty$.

4. (a) Define a bounded linear map $R: l^2 \to l^2$ defined by the formula

$$R(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, a_3, \ldots).$$

   What is the set of eigenvalues of R ?

   (b) Find the spectrum of R.

   (c) Define a bounded linear map $T: l^2 \to l^2$ by the formula

$$T(a_1, a_2, a_3, \ldots) = (a_1, a_2, a_3 - 2a_1, a_4 - 2a_2, a_5 - 2a_3, \ldots).$$

   Find the spectrum of T.

   [Hint: Use the previous part of the question and the spectral mapping theorem for polynomials]

5. Let A be a Banach algebra. Let $x \in A$ be an *involution*, that is to say an element where $x^2 = x$. Show that $Spectrum(x) \subseteq \{0, 1\}$.

Chapter 8

1. (a) Let H be a Hilbert space, and let $T: H \to H$ be a bounded linear map. Prove that the map T is unitary if and only if it is an isometric isomorphism.

   (b) Give an example of an isometry that is not a unitary.

2. Let H be a Hilbert space, and let $T: V \to V$ be a bounded linear map. Show that T is self-adjoint if and only if $\langle v, Tv \rangle \in \mathbb{R}$ for all $v \in H$.

3. (a) Let H be a Hilbert space, and let $T: H \to H$ be a bounded linear map. Show that the operator T^*T is positive.

   (b) Let $A: H \to H$ be a positive operator. Show that $Spectrum(A) \subseteq [0, \|A\|]$.

(c) Let $S: l^2 \to l^2$ be the shift operator defined by the formula

$$S(a_1, a_2, a_3, \ldots) = (a_2, a_3, a_4, \ldots).$$

Show that $\{0, 1\} \subseteq Spectrum(S^*S) \subseteq [0, 1]$.

4. Let $T: V \to W$ be a linear map between normed vector spaces. Show that T is compact if and only if for every bounded sequence (x_n), the image (Tx_n) has a convergent subsequence.

5. Define $T: l^2 \to l^2$ by

$$T(a_1, a_2, a_3, \ldots) = (a_1, \frac{a_2}{2}, \frac{a_3}{3}, \ldots).$$

Show that T is a compact self-adjoint operator.

6. Give an example of a self-adjoint bounded linear map on a Hilbert space without any eigenvalues.

7. Let $(e_n)_{n\in\mathbb{N}}$ be a basis of continuous functions for the Hilbert space $L^2[0, 1]$. For $m, n \in \mathbb{N}$, define a function $\phi_{m,n}: [0, 1] \times [0, 1] \to \mathbb{C}$ by the formula $\phi_{m,n}(s, t) = e_m(s)e_n(t)$. Prove that $(\phi_{m,n})_{m,n\in\mathbb{N}}$ is a basis for $L^2([0, 1] \times [0, 1])$.

8. Let $f: [0, 1] \to \mathbb{C}$ be a continuous function. Show that the following are equivalent for a continuous function $y: [0, 1] \to \mathbb{C}$:

   - y satisfies the differential equation

   $$y''(s) + f(s) = 0$$

   with boundary conditions

   $$y(0) = y(1) = 0.$$

   - $y = Kf$, where $K: L^2[0, 1] \to L^2[0, 1]$ is the compact operator with integral kernal defined by the formula

   $$k(s, t) = \begin{cases} t(1 - s) & 0 \le t \le s \\ s(1 - t) & s \le t \le 1 \end{cases}$$

9. Let $a(s, t) = \min(s, t) - st$. Let $A: L^2[0, 1] \to L^2[0, 1]$ be the bounded linear map defined by the formula

$$(Af)(s) = \int_0^1 a(s, t)f(t)\, dt.$$

(a) Find all non-zero eigenvalues and their associated eigenvectors.

(b) Find $\|A\|$.

Chapter 9

1. Let H be a Hilbert space, and let $T\colon H \to H$ be a bounded linear map. Prove that $T[H]^{\perp} = \ker T^*$.

2. (a) Let V be a normed vector space. Let $\alpha\colon V \to \mathbb{F}$ be a linear map. Show that α is continuous if and only if $\ker \alpha$ is a closed subspace of V.

   (b) Let V be a Banach space, let A be a closed subspace of V, and let B be a finite-dimensional subspace of V. Show that the subspace $A+B$ is closed.

3. (a) Let $T\colon l^2 \to l^2$ be defined by the formula

$$T(a_1, a_2, a_3, \ldots) = \left(\frac{a_1}{1}, \frac{a_2}{2}, \frac{a_3}{3}, \ldots\right).$$

   Prove that T is a bounded linear map, but the image of T is *not* closed.

   (b) Give an example of two closed subspaces, A and B, of a Hilbert space H, where the sum $A + B$ is not closed.

4. Let H be a Hilbert space.

   (a) Let $V\colon H \to H$ be a partial isometry with initial subspace M and final subspace N. Show that the maps V^*V and VV^* are the projections onto the subspaces M and N respectively.

   (b) Let M and N be closed subspaces of H. Let $V\colon H \to H$ be a bounded linear map such that V^*V and VV^* are the projections onto the subspaces M and N respectively.

   Show that V is a partial isometry with initial subspace M and final subspace N.

5. Let $T\colon H \to H$ be a linear map with finite-dimensional image.

   (a) Show that the trace $tr(T)$ is independent of choice of orthonormal basis for the space $T[H]$.

   (b) Let $S\colon H \to H$ be another linear map with finite-dimensional image. Show that $tr(ST) = tr(TS)$.

   (c) Let $P\colon H \to H$ be a projection onto a finite-dimensional subspace. Show that $tr(P) = \dim P[H]$.

6. (a) Let H_1 and H_2 be a Hilbert spaces, and let $T_1\colon H_1 \to H_1$ and $T_2\colon H_2 \to H_2$ be Fredholm operators. Show that the operator

$$(T_1 \oplus T_2)\colon H_1 \oplus H_2 \to H_1 \oplus H_2$$

defined by the formula

$$(T_1 \oplus T_2)(x+y) = T_1(x) + T_2(y) \qquad x \in H_1,\ y \in H_2$$

is Fredholm, with index $Ind(T_1 \oplus T_2) = Ind(T_1) + Ind(T_2)$.

(b) Let H be a Hilbert space, and let $S: H \to H$ be an invertible linear map and let $T: H \to H$ be a Fredholm operator. Prove that the composite ST is Fredholm, with $Ind(ST) = Ind(T)$.

7. Let H be a Hilbert space, and let $K: H \to H$ be a compact operator.

   (a) Let $\lambda \in \mathbb{C}\backslash\{0\}$. Show that either $\lambda I - K$ is invertible, or else λ is an eigenvalue of K with finite-dimensional eigenspace.
   [Hint: Let $F = I - \lambda^{-1}K$. Then F is a Fredholm operator with index zero. Look at what happens when $\ker F = \{0\}$, and when $\ker F \neq \{0\}$.]

   (b) Show that every point of the spectrum of K, apart possibly from 0, is an eigenvalue.

   (c) Show that the spectrum of K is countable, and the only possible limit of distinct points in the spectrum of K is zero.

8. Let $f: \mathbb{T} \to \mathbb{C}\backslash\{0\}$ be a continuous map. Using whatever results you need from the lecture notes (obviously, apart from theorem 4.26), prove that the Toeplitz operator T_f is Fredholm.

9. Calculate the index of the Toeplitz operator T_f for the following functions, $f: \mathbb{T} \to \mathbb{C}\backslash\{0\}$

   (a) $f(z) = z^3$ for $z \in \mathbb{T}$.

   (b) $f(z) = \exp(iz)$ for $z \in \mathbb{T}$.

   (c) $f(z) = \cos(2z)$ for $z \in \mathbb{T}$.

Chapter 10

1. (a) Let $p \geq 1$. Let $L^p_{\text{cont}}(\mathbb{R})$ be the space of continuous functions $f: \mathbb{R} \to \mathbb{C}$ such that the function $|f|^p$ is integrable. Show that we have a norm on $L^p_{\text{cont}}(\mathbb{R})$ defined by the formula

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

   (b) Prove that we have a well-defined inner product on the space $L^2_{\text{cont}}(\mathbb{R})$ defined by the formula

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \overline{f(t)} g(t) \, dt.$$

2. Let $f \in L^1(\mathbb{R})$ be piecewise-continuous. Let $\alpha \in \mathbb{R}$. Show that the following statements all hold.

   (a) $\mathfrak{F}\{e^{i\alpha x} f(x)\} = \hat{f}(\omega - \alpha)$.

   (b) $\mathfrak{F}\{f(x - x_0)\} = \hat{f}(\omega) e^{-i\omega x_0}$.

   (c) Let $\alpha > 0$. The $\mathfrak{F}\{f(\alpha x)\} = \hat{f}\left(\frac{\omega}{\alpha}\right)$.

   (d) Let $\alpha, \beta \in \mathbb{C}$ and g be another piecewise-continuous function in $L^1(\mathbb{R})$. Then $\mathfrak{F}\{\alpha f(x) + \beta g(x)\} = \alpha \hat{f}(\omega) + \beta \hat{g}(\omega)$.

3. Let $f \in L^1(\mathbb{R})$ be piecewise-differentiable. Suppose $f' \in L^1(\mathbb{R})$, and $\lim_{|x| \to \infty} f(x) = 0$. Prove that

$$\mathfrak{F}\{f'(x)\} = i\omega \hat{f}(\omega).$$

4. Let $\alpha > 0$. Calculate the Fourier transform $\mathfrak{F}(e^{-\alpha|x|})$.

5. Let $\sigma > 0$. Let
$$g_\sigma = \frac{1}{\sigma} \exp\left(\frac{-x^2}{2\sigma^2}\right).$$
   (a) Calculate the Fourier transform $\hat{g_\sigma}$.
   (b) Prove that for any $\sigma, \tau > 0$ we have
$$g_\sigma * g_\tau = g_{\sqrt{\sigma^2+\tau^2}}.$$

6. Let $f, g \in L^2_c(\mathbb{R})$. Prove that
$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)\ dx = \int_{-\infty}^{\infty} \hat{f}(x)g(x)\ dx.$$

7. Let ϕ be the function with Fourier transform $\hat{\phi} = \frac{1}{\sqrt{2\pi}}\chi_{[-\pi,\pi]}$. Show that the set $\{\phi_{0,k} \mid k \in \mathbb{Z}\}$ is orthonormal.

8. Let
$$H(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$
   (a) Calculate the Fourier transfrom $\hat{H}$.
   (b) Show that H is an admissible wavelet.

9. Let ψ be an admissible wavelet. Let $\phi \in L^1(\mathbb{R})$ be a bounded piecewise-continuous function. Show that the convolution $\psi * \phi$ is an admissible wavelet.

10. Let ψ be an admissible wavelet. Set
$$\psi_{s,t}(x) = \frac{1}{\sqrt{|s|}}\psi\left(\frac{x-t}{s}\right).$$
Show that
$$\hat{\psi}_{s,t}(\omega) = \sqrt{|s|}e^{-it\omega}\hat{\psi}(s\omega).$$