Functional Analysis: Problem Booklet

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SEMESTER ONE

Chapter 1

1. Let V be a normed vector space.

   (a) Prove that
   $$\|x - y\| \geq |\|x\| - \|y\||$$
   for all vectors $x, y \in V$.

   (b) Prove that the norm defines a continuous map $V \to \mathbb{R}^{\geq 0}$ by writing $x \mapsto \|x\|$.

   (c) In a normed vector space V prove that addition defines a continuous function $+ \colon V \times V \to V$. What can you say about scalar multiplication?

2. Let V and W be normed vector spaces. Let us write $V \oplus W$ to denote the space of pairs
   $$V \oplus W = \{(x, y) \mid x \in V, y \in W\}.$$

   (a) Show that we can define a norm on the space $V \oplus W$ by the formula
   $$\|(x, y)\| = \|x\| + \|y\| \qquad x \in V,\ y \in ".$$

   (b) Let (x_n) and (y_n) be sequences in the spaces V and W with norm limits x and y respectively. Show that the sequence (x_n, y_n) in $V \oplus W$ converges in norm in $V \oplus W$ to the limit (x, y).

   (c) Suppose that V and W are Banach spaces. Is $V \oplus W$ necessarily a Banach space? Justify your answer.

   (d) Let $n \in \mathbb{N}$. Use the above to prove that for $\mathbb{R}^n$ and $\mathbb{C}^n$ are Banach spaces. You may use without proof the fact that $\mathbb{R}$ and $\mathbb{C}$ are complete.

3. (a) Let c_0 be the vector space of sequences (a_n) in $\mathbb{C}$ such that $a_n \to 0$ as $n \to \infty$, with pointwise addition and scalar multiplication. Prove that we have a norm on c_0 defined by the formula

$$\|(a_n)\| = \sup\{|a_n| \mid n \in \mathbb{N}\}.$$

   (b) Is the space c_0 a Banach space? Justify your answer.

4. Let l^∞ be the Banach space of bounded sequences, (a_n), in $\mathbb{C}$, with norm defined as above. Which of the following are closed subsets of l^∞? Justify your answer.

   (a) The space c_0.

   (b) The vector space c_{00} of all sequences (a_n) of complex numbers for which there exists N with $a_n = 0$ whenever $n \geq N$.

   (c) For a given N, the vector space c_N of all sequences (a_n) of complex numbers such that $a_n = 0$ whenever $n \geq N$.

Chapter 2

1. Define linear maps $S, T \colon C[0,1] \to C[0,1]$ by the formulae

$$S(f)(s) = \int_0^s f(t)\ dt \qquad T(g)(s) = sg(s).$$

   (a) Prove that S and T are bounded, and find $\|S\|$ and $\|T\|$.

   (b) Do S and T commute? Find $\|ST\|$ and $\|TS\|$.

2. Let P be the set of all polynomial functions.

   (a) Show that P has a norm defined by the formula

$$\|a_0 + a_1x + \cdots + a_nx^n\| = \max(|a_1|, \ldots, |a_n|).$$

   (b) Let $A \colon P \to P$ and $B \colon P \to P$ be linear maps defined by the formulae

$$A(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = a_0 + a_1x + 2a_2x^2 + \cdots + na_nx^n$$

   and

$$B(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = a_0 + a_1x + \frac{1}{2}a_2x^2 + \cdots + \frac{1}{n}a_nx^n$$

   respectively.

   Show that B is invertible, with inverse A.

   (c) Show that B is a bounded linear map. Is A a bounded linear map? Justify your answer.

(d) From the above and the open mapping theorem, what can we deduce about the completeness of P ?

3. Let $C^\infty[0,1]$ be the space of functions $f\colon[0,1]\to\mathbb{C}$ such that derivatives of all orders exist.

   (a) Show that $C^\infty[0,1]$ has a norm defined by the formula

   $$\|f\| = \sup\{|f(t)| \mid t\in[0,1]\}.$$

   (b) Let

   $$C_0^\infty[0,1] = \{f\in C^\infty[0,1] \mid f(0)=0\}$$

   Show that $C_0^\infty[0,1]$ is a closed subspace of $C^\infty[0,1]$.

   (c) Define $I\colon C^\infty[0,1]\to C_0^\infty[0,1]$ by the formula

   $$I(f)(x) = \int_0^x f(t)\ dt$$

   Prove that I is a bounded linear map, and is bijective.

   (d) What is the inverse of I ? Is the inverse a bounded linear map ?

   (e) What can we deduce about the completeness of $C^\infty[0,1]$?

4. Let V and W be normed vector spaces, and let $T\colon V\to W$ be a surjective linear map.

   (a) Prove that T is open if and only if there exists $m>0$ such that

   $$\|Tv\|\geq m\|v\|$$

   for all $v\in V$.

   (b) Prove that T is open if and only if there exists $\delta>0$ such that

   $$B_W(0,\delta)\subseteq T[B_V(0,1)]\subseteq W.$$

   (c) Give an example of a bounded and bijective linear map between normed vector spaces which is not open.

Chapter 3

1. Let V and W be a normed vector space. Use the Hahn-Banach theorem to show the following statements are true.

   (a) Let $V\neq\{0\}$. Then $V^*\neq\{0\}$.

   (b) Let $x,y\in V$. Suppose $f(x)=f(y)$ for every bounded linear functional f. Then $x=y$.

(c) Let $c \geq 0$ and $x \in V$. Suppose $|f(x)| \leq c$ whenever $f \in V^*$ and $\|f\| \leq 1$. Then $\|x\| \leq c$.

2. Let V be a vector space, and let W be a subspace of V. Prove using Zorn's lemma that there is a vector space Z such that $Z \cap W = \{0\}$ and $Z + W = V$.

3. Let A be the set of all functions $f\colon [0, \pi] \to \mathbb{R}$ of the form

$$f(x) = \alpha_0 + \alpha_1 \cos x + \cdots + \alpha_n \cos(nx)$$

where $\alpha_i \in \mathbb{R}$.

Prove that A is a dense subset of the set $C_{\mathbb{R}}[0, \pi]$.

4. Let V be a normed vector space, let $\{v_1, \ldots, v_n\}$ be linearly independendent, and let $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$. Prove that we have a bounded linear map $f\colon V \to \mathbb{F}$ such that $f(v_i) = \lambda_i$ for all i.

Chapter 4

1. Let H be a Hilbert space, and let S be a subset of H.

   (a) Prove that $S^\perp$ is a linear subspace of H, and that $S^\perp = Span(S)^\perp$.

   (b) If $x \in S$ and $v \in S^\perp$, show that $\|x + v\|^2 = \|x\|^2 + \|v\|^2$.

   (c) Let A and B be subsets of H. Prove that

   $$(A \cup B)^\perp A^\perp \cap B^\perp.$$

   (d) Let $S' \subseteq S$. Prove that $S^\perp \subseteq (S')^\perp$.

   (e) Let V be a subspace of H. Prove that $(\overline{V})^\perp = V^\perp$.

2. Let H and H' be Hilbert spaces, and let $T\colon H \to H$ be a bounded linear map. Show that

$$T[H]^\perp = \ker T^*.$$

3. (a) Let H be a Hilbert space. Prove that we have a conjugate-linear isometry $J\colon H \to H^*$ defined by writing

   $$J(v)(x) = \langle v, x \rangle \qquad v \in H,\ x \in H.$$

   (b) Prove that the map J is invertible. You may use the Riesz representation theorem without proof.

   (c) Let H be a Hilbert space. Prove that the spaces H and $(H^*)^*$ are isometrically isomorphic.

4. Define bounded linear maps $A, B\colon l^2 \to l^2$ by the formulae

$$A(a_1, a_2, a_3, a_4 \ldots) = (0, a_1, \frac{a_2}{2}, \frac{a_3}{3}, \frac{a_4}{4}, \ldots)$$

and

$$B(a_1, a_2, a_3, a_4 \ldots) = (a_2, \frac{a_1}{2}, a_4, \frac{a_3}{2}, a_6, \frac{a_5}{2}, \ldots)$$

respectively. Compute A^* and B^*.

Chapter 5

1. (a) What does it mean to say an infinite set is linearly independent ?

   (b) Prove that any orthogonal set is linearly independent.

2. Prove that every Hilbert space has a basis.

   [Hint: You need to use Zorn's lemma]

3. (a) Let S be a subset of a Hilbert space H. Prove that $S^{\perp} = Span(S)^{\perp}$.

   (b) Let H be a Hilbert space, and let $(e_n)_{n=1}^{\infty}$ be an orthonormal set such that for any element $x \in H$, we have

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle e_n, x\rangle|^2.$$

   Prove that (e_n) is an orthonormal basis of H.

4. Let $e_k(t) = \exp(ikt)$.

   (a) Let $f(x) = x$. Find coefficients $a_k \in \mathbb{C}$ such that

$$\sum_{k=-\infty}^{\infty} a_k e_k = f$$

   in the space $L^2[0, 2\pi]$.

   (b) Use the above to evaluate the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

SEMESTER TWO

Chapter 1

1. Let $f \in L^1(\mathbb{R})$ be piecewise-continuous. Let $\alpha \in \mathbb{R}$. Show that the following statements all hold.

   (a) $\mathfrak{F}\{e^{i\alpha x} f(x)\} = \hat{f}(\omega - \alpha)$.

   (b) $\mathfrak{F}\{f(x - x_0)\} = \hat{f}(\omega)e^{-i\omega x_0}$.

   (c) Let $\alpha > 0$. The $\mathfrak{F}\{f(\alpha x)\} = \hat{f}\left(\frac{\omega}{\alpha}\right)$.

   (d) Let $\alpha, \beta \in \mathbb{C}$ and g be another piecewise-continuous function in $L^1(\mathbb{R})$. Then $\mathfrak{F}\{\alpha f(x) + \beta g(x)\} = \alpha\hat{f}(\omega) + \beta\hat{g}(\omega)$.

   (e) Let $f \in L^1(\mathbb{R})$ be piecewise-differentiable. Suppose $f' \in L^1(\mathbb{R})$, and $\lim_{|x|\to\infty} f(x) = 0$. Prove that

   $$\mathfrak{F}\{f'(x)\} = i\omega\hat{f}(\omega).$$

2. Let $\alpha > 0$. Calculate the Fourier transform $\mathfrak{F}(e^{-\alpha|x|})$.

3. Let $\sigma > 0$. Let

   $$g_\sigma = \frac{1}{\sigma}\exp\left(\frac{-x^2}{2\sigma^2}\right).$$

   (a) Calculate the Fourier transform $\hat{g_\sigma}$.

   (b) Prove that for any $\sigma, \tau > 0$ we have

   $$g_\sigma * g_\tau = g_{\sqrt{\sigma^2+\tau^2}}.$$

4. Let ϕ be the function with Fourier transform $\hat{\phi} = \frac{1}{\sqrt{2\pi}}\chi_{[-\pi,\pi]}$. Show that the set $\{\phi_{0,k} \mid k \in \mathbb{Z}\}$ is orthonormal.

Chapter 2

1. Let $k \in \mathbb{Z}$. Define a path $\gamma_k \colon [0,1] \to \mathbb{C}\backslash\{0\}$ by the formula

   $$\gamma_k(t) = \exp(2\pi i k t).$$

   (a) Let $U \subseteq \mathbb{C}$ be an open set containing the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$. Let $f\colon U \to \mathbb{C}$ be holomorphic. Show that

   $$\oint_{\gamma_k} f(z)\, dz = k \oint_{\gamma_1} f(z)\, dz.$$

   (b) Calculate $Wind(\gamma_k, a)$ in each of the two cases $|a| < 1$ and $|a| > 1$.

2. Write down Cauchy's formula for derivatives.

   [Note: This formula is *not* in the notes; the exercise is to find out what it is.]

3. (a) Let $0 < r < R$, and let $f\colon B(0,R) \to \mathbb{C}$ be a holomorphic function. Let $M = \sup\{|f(z)| \mid |z| = r\}$. Use Cauchy's formula for derivatives to show that
   $$|f'(w)| \leq \frac{Mr}{(r-|w|)^2}$$
   for all $w \in B(0,r)$.

   (b) Use the above to prove Liouville's theorem.

4. (a) Let $p\colon \mathbb{C} \to \mathbb{C}$ be a non-constant polynomial (with complex coefficients). Prove that $p(z) \to \infty$ as $z \to \infty$.

   (b) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a function. Prove that $\lim_{z\to\infty} f(z) = \infty$ if and only if $\lim_{z\to\infty} \frac{1}{f(z)} = 0$.

   (c) Use the above, and Liouville's theorem to prove the *fundamental theorem of algebra*, which states that for every non-constant complex polynomial $p\colon \mathbb{C} \to \mathbb{C}$, there is at least one point $z_0 \in \mathbb{C}$ for which $p(z_0) = 0$.

Chapter 3

1. Let A be a unital Banach algebra.

   (a) Let $x \in A$, and $\lambda \in \mathbb{F}$. Suppose that $\|x\| < |\lambda|$. Show that the element $\lambda - x$ is invertible.

   (b) Show that the set of invertible elements in A is open.

2. Let $f\colon [0,1] \to \mathbb{C}$ be a continuous map. Define $M_f\colon C[0,1] \to C[0,1]$ by the formula $M_f(g)(t) = f(t)g(t)$. Prove that the operator M_f is a bounded linear map. and find its spectrum.

3. (a) Define a bounded linear map $R\colon l^2 \to l^2$ defined by the formula
   $$R(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, a_3, \ldots).$$
   What is the set of eigenvalues of R ?

   (b) Find the spectrum of R.

   (c) Define a bounded linear map $T\colon l^2 \to l^2$ by the formula
   $$T(a_1, a_2, a_3, \ldots) = (a_1, a_2, a_3 - 2a_1, a_4 - 2a_2, a_5 - 2a_3, \ldots).$$
   Find the spectrum of T.

   [Hint: Use the previous part of the question and the spectral mapping theorem for polynomials]

Chapter 4

1. (a) Let H be a Hilbert space, and let $T: H \to H$ be a bounded linear map. Prove that the map T is unitary if and only if it is an isometric isomorphism.

   (b) Give an example of an isometry that is not a unitary.

2. Let H be a Hilbert space, and let $T: V \to V$ be a bounded linear map. Show that T is self-adjoint if and only if $\langle v, Tv \rangle \in \mathbb{R}$ for all $v \in H$.

3. Let $T: V \to W$ be a linear map between normed vector spaces. Show that T is compact if and only if for every bounded sequence (x_n), the image (Tx_n) has a convergent subsequence.

4. Define $T: l^2 \to l^2$ by

$$T(a_1, a_2, a_3, \ldots) = (a_1, \frac{a_2}{2}, \frac{a_3}{3}, \ldots).$$

   Show that T is a compact self-adjoint operator.

Chapter 5

1. Let H be a Hilbert space, and let $T: H \to H$ be a bounded linear map. Prove that $T[H]^\perp = \ker T^*$.

2. (a) Let V be a normed vector space. Let $\alpha: V \to \mathbb{F}$ be a linear map. Show that α is continuous if and only if $\ker \alpha$ is a closed subspace of V.

   (b) Let V be a Banach space, let A be a closed subspace of V, and let B be a finite-dimensional subspace of V. Show that the subspace $A + B$ is closed.

3. Let $T: l^2 \to l^2$ be defined by the formula

$$T(a_1, a_2, a_3, \ldots) = \left(\frac{a_1}{1}, \frac{a_2}{2}, \frac{a_3}{3}, \ldots \right).$$

   Prove that T is a bounded linear map, but the image of T is *not* closed.

4. Calculate the index of the Toeplitz operator T_f for the following functions, $f: \mathbb{T} \to \mathbb{C} \backslash \{0\}$

   (a) $f(z) = z^3$ for $z \in \mathbb{T}$.

   (b) $f(z) = \exp(iz)$ for $z \in \mathbb{T}$.

   (c) $f(z) = \cos(2z)$ for $z \in \mathbb{T}$.