Descent and the KH-assembly map

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Contents

Abstract

In this article we show that a general notion of descent in coarse geometry can be applied to the study of injectivity of the KH-assembly map. We also show that the coarse assembly map is injective in general for finite coarse CW-complexes.

1 Introduction

In [20], the author drew a general picture of the principle of descent from isomorphism conjectures in coarse geometry to the injectivity of assembly maps in the sense of [6, 28]. This technique was applied to the analytic Novikov conjecture and the algebraic K-theory assembly map.

Our main purpose in this article is to show how homotopy algebraic K-theory fits into this picture. We also look at the stable versions of bivariant algebraic K-theory developed in [7, 8] to provide versions of the KH-isomorphism conjecture with coefficients.

We need some machinery to carry out these procedures. Firstly, there is the general descent principle already mentioned. In this article we give a new result saying that *any* coarse assembly map is an isomorphism for finite coarse CW-complexes. It is an interesting question whether this could be extended, for example along the lines of [29, 30] to show that any coarse assembly map is an isomorphism for spaces with finite asymptotic dimension. However, that is a project for another time.

Back to the present, we also use the bivariant algebraic K-theory spectra of [8]. These are defined for algebras; for our purposes we extend the construction to algebroids using the same techniques as used by Joachim to define the K-theory of C^*-categories in [11].

The properties of bivariant algebraic K-theory spectra are then applied to prove that a certain construction along the lines of that in [1] yields a coarse homology theory in the sense of [19]. The application of the descent machinery to the maps under consideration then follows.

2 Coarse Assembly

We begin by recalling some machinery from coarse geometry. First, recall that a *coarse space* is a set X along with a collection of priveleged subsets $M \subseteq X \times X$ called *controlled sets*. The collection of controlled sets is called a *coarse structure* on X, and is required to satisfy certain axioms.

To be precise, the collection of controlled sets is required to contain the diagonal, $\Delta_X \subseteq X \times X$.[1] The controlled sets must be closed under finite unions, taking subsets, reflections in the diagonal, and composition. Here, by the *composition* of subsets $M_1, M_2 \subseteq X \times X$, we mean the set

$$M_1 M_2 = \{(x,z) \in X \times X \mid (x,y) \in M_1 \ (y,z) \in M_2 \text{ for some } y \in X\}.$$

We refer the reader to [22], for example, for further details.

Given a controlled set $M \subseteq X \times X$, and a subset $S \subseteq X$, we write

$$M[S] = \{y \in X \mid (x,y) \in M \text{ for some } x \in S\}$$

For a point $x \in X$, we write $M_x = M[\{x\}]$.

If X is a coarse space, and $f, g \colon S \to X$ are maps into X, the maps f and g are termed *close* or *coarsely equivalent* if the set $\{(f(s), g(s)) \mid s \in S\}$ is controlled. We call a subset $B \subseteq X$ *bounded* if the inclusion $B \hookrightarrow X$ is close to a constant map, or equivalently $B = M_x$ for some controlled set M and some point $x \in X$.

We also need the notion of a topological space with a compatible coarse structure. To be precise, if X is a Hausdorff space, we call it a *coarse topological space* if it has a coarse structure where every controlled set is contained in an open controlled set (with the usual product topology on $X \times X$), and the closure of any bounded set is compact.

If X is a coarse topological space, we say the coarse structure is *compatible* with the topology.

The most important examples of coarse spaces to us are the following. The first is standard.

[1] This axiom is dropped for *non-unital* coarse spaces; see [15].

Example 2.1 If X is a proper metric space. Equip X with a coarse structure defined by defining the controlled sets to be subsets of *neighbourhoods of the diagonal*:

$$N_R = \{(x, y) \in X \times X \mid d(x, y) < R\}$$

Then X is a coarse topological space.

Our second example comes from [20].

Example 2.2 Let X be a coarse topological space. Suppose that X is a topologically dense subset of a Hausdorff space $\overline{X}$. Call the coarse structure already defined on the space X the *ambient coarse structure*, and set $\partial X = \overline{X} \backslash X$.

Call an open subset $M \subseteq X \times X$ *strongly controlled* if:

- The set M is controlled with respect to the ambient coarse structure on X.
- Let $\overline{M}$ be the closure of the set M in the space $\overline{X}$. Then $\overline{M} \cap (\overline{X} \times \partial X) \cup (\partial X \times \overline{X})$ is contained in the diagonal of ∂X.

We define the *continuously controlled coarse structure* with respect to $\overline{X}$ by saying that the controlled sets are composites of subsets of strongly controlled open sets.

We write X to denote the space X with its ambient coarse structure, and X^{cc} to denote the space X with the new continuously controlled coarse structure.

It is shown in [20] that X^{cc} is a coarse topological space. Our next few examples are certain standard constructions of coarse spaces.

Example 2.3 Let X be a coarse space, let $\sim$ be an equivalence relation on X, and let $X/\sim$ be the set of equivalence classes. Let $\pi \colon X \to X/\sim$ be the quotient map sending each point $x \in X$ to its equivalence class, $\pi(x)$.

We define the *quotient coarse structure* on $X/\sim$ by saying a subset $M \subseteq X/\sim \times X/\sim$ is controlled if and only if $M = \pi[M']$ for some controlled set $M' \subseteq X \times X$.

Example 2.4 Let X and Y be coarse spaces. Then we define the *product*, $X \times Y$ to be the Cartesian product of the sets X and Y equipped with the coarse structure defined by saying a subset $M \subseteq (X \times Y) \times (X \times Y)$ is controlled if and only if we have controlled sets $M_1 \subseteq X \times X$ and $M_2 \subseteq Y \times Y$ such that

$$M \subseteq \{(u, v, x, y) \mid (u, x) \in M_1,\ (v, y) \in M_2\}.$$

Example 2.5 Let $\{X_i \mid i \in I\}$ be a collection of coarse spaces. Then, as a set, the *coarse disjoint union*, $\vee_{i \in I} X_i$ is the disjoint union of the sets X_i.

A subset $M \subseteq (\vee_{i \in I} X_i) \times (\vee_{i \in I} X_i)$ is controlled if it is a subset of a union of the form

$$\left(\bigcup_{i \in I} M_i \right) \cup \left(\bigcup_{i, j \in I} B_i \times B_j \right)$$

where each set $M_i \subseteq X_i \times X_i$ is controlled, and each $B_j \subseteq X_j$ is bounded.

The following definition comes from [17].

Definition 2.6 Let R be the topological space $[0, \infty)$ equipped with a coarse structure compatible with the topology. We call the space R a *generalised ray* if the following conditions hold.

- Let $M, N \subseteq R \times R$ be controlled sets. Then the sum

$$M + N = \{(u + x, v + y) \mid (u, v) \in M,\ (x, y) \in N\}$$

  is controlled.

- Let $M \subseteq R \times R$ be a controlled set. Then the set

$$M^s = \{(u, v) \in R \times R \mid x \leq u, v \leq y,\ (x, y) \in M\}$$

  is controlled.

- Let $M \subseteq R \times R$ be a controlled set, and $a \in R$. Then the set

$$a + M = \{(a + x, a + y) \mid (x, y) \in M\}$$

  is controlled.

The classic example of a generalised ray is the metric space $\mathbb{R}_+$ equipped with the bounded coarse structure. There are others.

Now, a map between coarse spaces is called a *coarse map* if the image of a controlled set is controlled, and the pre-image of a bounded set is bounded. The *coarse category* is the category of all coarse spaces and coarse maps.

We call a coarse map $f\colon X \to Y$ a *coarse equivalence* if there is a coarse map $g\colon Y \to X$ such that the composites $g \circ f$ and $f \circ g$ are close to the identities 1_X and 1_Y respectively.

Coarse spaces X and Y are said to be *coarsely equivalent* if there is a coarse equivalence between them.

The notion of *coarse homotopy* is a weakening of this notion. To be precise, we have the following.

Definition 2.7 Let X be a coarse space equipped with a coarse map $p\colon X \to R$. Then we define the *p-cylinder of X*:

$$I_pX = \{(x, t) \in X \times R \mid t \leq p(x) + 1\}$$

We define coarse maps $i_0, i_1\colon X \to I_pX$ by the formulae $i_0(x) = (x, 0)$ and $i_1(x) = (x, p(x) + 1)$ respectively.

Definition 2.8 Let $f_0, f_1 \colon X \to Y$ be coarse maps. An *elementary coarse homotopy* between f_0 and f_1 is a coarse map $H \colon I_p X \to Y$ for some $p \colon X \to R$ such that $f_0 = H \circ i_0$ and $f_1 = H \circ i_1$.

More generally, we call the maps f_0 and f_1 *coarsely homotopic* if they can be linked by a finite sequence of elementary coarse homotopies.

Note that close coarse maps are always coarsely homotopic.

A coarse map $f \colon X \to Y$ is termed a *coarse homotopy equivalence* if there is a coarse map $g \colon Y \to X$ such that the compositions $g \circ f$ and $f \circ g$ are coarsely homotopic to the identities 1_X and 1_Y respectively.

Before we introduce the main piece of machinery from [20], we need two more technical notions.

Definition 2.9 Let X be a coarse space. Then we call a decomposition $X = A \cup B$ *coarsely excisive* if for every controlled set $m \subseteq X \times X$ there is a controlled set $M \subseteq X \times X$ such that $m(A) \cap m(B) \subseteq M(A \cap B)$.

Definition 2.10 We call a coarse space X *flasque* if there is a map $\tau \colon X \to X$ such that:

- Let $B \subseteq X$ be bounded. Then there exists $N \in \mathbb{N}$ such that $\tau^n[X] \cap B = \emptyset$ for all $n \geq N$.
- Let $M \subseteq X \times X$ be controlled. Then the union $\bigcup_{n \in \mathbb{N}} \tau^n[M]$ is controlled.
- The map τ is close to the identity map.

A generalised ray is clearly flasque. We are now ready for the main construction of this section.

Definition 2.11 We call a functor, E, from the coarse category to the category of spectra *coarsely excisive* if the following conditions hold.

- The spectrum $E(X)$ is weakly contractible whenever the coarse space X is flasque.
- The functor E takes coarse homotopy equivalences to weak homotopy equivalences of spectra.
- For a coarsely excisive decomposition $X = A \cup B$ we have a homotopy push-out diagram
$$\begin{array}{ccc} E(A \cap B) & \to & E(A) \\ \downarrow & & \downarrow \\ E(B) & \to & E(X) \end{array} .$$

Now, let X be a coarse topological space. Then we define the *open square*, $\mathcal{S}X$, to be the space $X \times [0,1)$ equipped with the continuously controlled coarse structure arising from considering $\mathcal{S}X$ as a dense subset of $X \times [0,1]$. It is shown in [20] that the open square $\mathcal{S}X$ is always flasque.

We define the *open* and *closed cones* to be the quotients

$$\mathcal{O}X = \frac{\mathcal{S}X}{X \times \{0\}} \qquad \mathcal{C}X = \frac{X \times [0,1]}{X \times \{0\}}$$

respectively.

The following result also comes from [20].

Lemma 2.12 *Let X be a coarse topological space, and let E be a coarsely excisive functor. Then we have a natural weak fibration of spectra*

$$E(X) \to E(\mathcal{C}X) \to E(\mathcal{O}X).$$

Further, if the space X is flasque, then so is the closed cone $\mathcal{C}X$. □

Definition 2.13 The boundary map $\partial\colon \Omega E(\mathcal{O}X) \to E(X)$ of the above weak fibration is called the *coarse assembly map* associated to E.

Thus the map ∂ is a weak equivalence if and only if the space $E(\mathcal{C}X)$ is weakly contractible. As shown in [20], if the space X is flasque, then so is the closed cone $\mathcal{C}X$. Thus the map ∂ is a weak equivalence whenever X is flasque.

Our aim in the rest of this section is to generalise this observation.

Definition 2.14 Let X and Y be coarse spaces. Let $A \subseteq X$, and let $f\colon A \to Y$ be a coarse map. Then we define the space obtained by *attaching* X to Y *along* A by the map f to be the quotient $X \cup_A Y = X \vee Y/{\sim}$, where $\sim$ is the equivalence relation defined by saying $a \sim f(a)$ whenever $a \in A$.

Now, let R be a generalised ray. We define the *coarse R-disk* and *coarse R-sphere* of dimensions n and $n-1$ respectively to be the spaces

$$D^n_R = (R \vee R)^n \times R \qquad S^n_R = (R \vee R)^{n+1} \times \{0\}.$$

Note that $S^n_R \subseteq D^{n+1}_R$. It is shown in [21] that the coarse disk D^n_R and generalised ray R are coarsely homotopy-equivalent.[2] Given a coarse map $f\colon S^n_R \to Y$, we can form the coarse space $D^{n+1}_R \cup_{S^n_R} Y$.

Lemma 2.15 *Let $\pi\colon D^{n+1}_R \vee Y \to D^{n+1}_R \cup_{S^n_R} Y$ be the quotient map. Then we have a coarsely excisive decomposition $D^{n+1}_R \cup_{S^n_R} Y = \pi[D^{n+1}_R] \cup \pi[Y]$, and the spaces $\pi[D^{n+1}_R]$ and $\pi[Y]$ are coarsely equivalent to the spaces D^{n+1}_R and Y respectively. Further, the intersection $\pi[D^{n+1}_R] \cap \pi[Y]$ is coarsely equivalent to the coarse sphere S^n_R.*

Proof: Apart from the comment about the intersection, this result follows from proposition 5.5 of [17], though it is not shown explicitly in [17] that the spaces $\pi[D^{n+1}_R]$ and D^{n+1}_R are coarsely equivalent. So we do this step here.

[2] There is a similar result in [17] for a more primitive notion of homotopy.

Observe that the spaces D_R^{n+1} and

$$(D_R^{n+1})' = \{(x,t) \in (R \vee R)^n \times R \mid t \geq 1\}$$

are coarsely equivalent. Now, maps which preserve controlled sets, such as π, preserve coarsely equivalent spaces, so the spaces $\pi[D_R^{n+1}]$ and $\pi[(D_R^{n+1})']$ are coarsely equivalent.

But by definition, the map $\pi|_{(D_R^{n+1})'} \colon (D_R^{n+1})' \to D_R^{n+1} \cup_{S_R^n} Y$ is a coarse equivalence onto its image. To summarise, we have a chain of coarse equivalences

$$D_R^{n+1} \sim (D_R^{n+1})' \sim \pi[(D_R^{n+1})'] \sim \pi[D_R^{n+1}].$$

As for the statement about intersections, the spaces $S_R^n = (R \vee R)^{n+1} \times \{0\}$ and $(R \vee R)^{n+1} \times \{1\}$ are certainly coarsely equivalent. Therefore the images $\pi[S_R^n]$ and $\pi[(R \vee R)^{n+1} \times \{1\}$ are coarsely equivalent.

But the space $\pi[(R \vee R)^{n+1} \times \{1\}$ is coarsely equivalent to S_R^n, and $\pi[S_R^n] = \pi[D_R^{n+1}] \cap \pi[Y]$. It follows that the spaces S_R^n and $\pi[D_R^{n+1}] \cap \pi[Y]$ are coarsely equivalent. □

The following also comes from [17].

Definition 2.16 We call a coarse space X a *finite coarse CW-complex* if we have subsets

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n = X$$

where:

- The space X_0 is a finite disjoint union of generalised rays.
- The space X_n is obtained from X_{n-1} by attaching finitely many coarse n-dimensional disks along coarse $(n-1)$-dimensional spheres.

We now come to the promised major result of this section.

Theorem 2.17 *Let E be a coarsely excisive functor, and let X be a finite coarse CW-complex. Then the coarse assembly map associated to E is a weak equivalence.*

Proof: We will show that $E(\mathcal{C}X)$ is weakly contractible.

Suppose we have a decomposition

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n = X$$

as in the above definition.

First, we know that a generalised ray is flasque, and it is easy to check that a finite coarse disjoint union of flasque spaces is flasque. Hence X_0 is flasque. By lemma 2.12, the closed cone of a flasque space is flasque, so by definition of an excisive functor, $E(\mathcal{C}X_0)$ is contractible.

We now work by induction. It suffices to show that if $E(\mathcal{C}Y)$ is weakly contractible, and we have a map $f\colon S_R^{k-1} \to Y$, then the spectrum $E(\mathcal{C}(D_R^k \cup_{S_R^{k-1}Y} Y))$ is also weakly contractible.

By lemma 2.15 and the homotopy pushout axiom in the definition of a coarsely excisive functor, we have a long exact sequence of stable homotopy groups

$$\pi_n E(\mathcal{C}D_R^k) \oplus \pi_n E(\mathcal{C}Y) \to \pi_n E(\mathcal{C}(D_R^k \cup_{S_R^{k-1}Y} Y)) \to \pi_{n-1} E(\mathcal{C}S_R^{k-1}) \to \pi_{n-1} E(\mathcal{C}D_R^k) \oplus \pi_{n-1} E(\mathcal{C}Y).$$

Now we know that $E(\mathcal{C}Y)$ is weakly contractible. The coarse space D_R^k is coarsely homotopy-equivalent to the flasque space R, so by lemma 2.12, the space $E(\mathcal{C}D_R^k)$ is weakly contractible. It therefore suffices to prove that the spectrum $E(\mathcal{C}S_R^m)$ is weakly contractible for each $m \in \mathbb{N}$.

The space S_R^0 is a coarse disjoint union of two generalised rays, so $E(\mathcal{C}S_R^0)$ is weakly contractible as argued above. We now work again by induction. Suppose the spectrum $E(\mathcal{C}S_R^p)$ is weakly contractible; we need to show that $E(\mathcal{C}S_R^{p+1})$ is weakly contractible.

Observe that we have a coarsely excisive decomposition $S_R^{p+1} = A \cup B$ where A and B are both coarsely equivalent to the coarse disk D_R^{p+1}, and $A \cap B$ is coarsely equivalent to the coarse sphere one dimension lower, S_R^p. Then it follows from an argument similar to the exact sequence argument three paragraphs above that the spectrum $E(S_R^{p+1})$ is weakly contractible, and we are done. $\square$

3 Bivariant Algebraic K-theory

Let R be a commutative ring with identity, and let A be an R-algebra. Let $A[x]$ be the algebra of polynomials over A in one variable. Define algebra homomorphisms $\partial_0, \partial_1 \colon A[x] \to A$ by writing $\partial_i(f) = f(i)$.

As in [9], given algebra homomorphisms $\alpha, \beta\colon A \to B$, we call an algebra homomorphism $H\colon A \to B[x]$ an *elementary algebraic homotopy* from α to β if $\partial_0 \circ H = \alpha$ and $\partial_1 \circ H = \beta$. More generally, we call α and β *algebraically homotopic* if they can be linked by a finite chain of algebraic homotopies, and write $\alpha \simeq \beta$.

Algebraic K-theory is *not* invariant under algebraic homotopies. However, in [12, 13, 26], the definition of algebraic K-theory is modified to define a series of groups, $KH_n(A)$, called *homotopy algebraic K-theory groups*.

In another world, that of complex C^*-algebras, there is a well-established notion of *bivariant K-theory groups*, $KK_n(A, B)$, defined for C^*-algebras A and B; see [14], or [10, 24] for overviews. These bivariant K-theory groups generalise ordinary K-theory in that $KK_n(\mathbb{C}, A) = K_n(A)$ for any C^*-algebra A.

Now, in [5], and independently in [7], there are constructions of *bivariant algebraic K-theory* groups $kk_n(A, B)$ for R-algebras A and B. These groups have certain universal properties that ensure they are isomorphic. To be completely explicit, the definition of bivariant algebraic K-theory that we are using here is the matrix-stable theory from section 9 of [7]. The article [8] represents this bivariant K-theory by spectra. Specifically, we have the following result.

Theorem 3.1 *Let $\mathcal{R}$ be the category of R-algebras. Let Sp be the category of spectra. We have a functor $\mathbb{KK}\colon \mathcal{R}^{\mathrm{op}} \times \mathcal{R} \to Sp$ with the following properties.*

- *Let A, B and C be R-algebras. Let $\alpha, \beta\colon B \to C$ be algebraically homotopic. Then the induced maps $\alpha_*, \beta_*\colon \mathbb{KK}(A,B) \to \mathbb{KK}(A,C)$ are homotopic, and the induced maps $\alpha^*, \beta^*\colon \mathbb{KK}(C,A) \to \mathbb{KK}(C,B)$ are homotopic.*

- *Let*
$$0 \to A \stackrel{i}{\to} B \stackrel{j}{\to} C \to 0$$
be a split exact short exact sequence of R-algebras. Let D be another R-algebra. Then we have induced homotopy fibrations of spectra
$$\mathbb{KK}(D,A) \stackrel{i_*}{\to} \mathbb{KK}(D,B) \stackrel{j_*}{\to} \mathbb{KK}(D,C) \to 0$$
and
$$\mathbb{KK}(C,D) \stackrel{j^*}{\to} \mathbb{KK}(B,D) \stackrel{i^*}{\to} \mathbb{KK}(A,D) \to 0.$$

- *Let A and B be R-algebras. Let $M_n(R)$ be the algebra of $n \times n$ matrices with values in R, and let $M_\infty(R)$ be the direct limit of the sequence of matrix algebras $(M_n(R))$ under the top-left inclusions $r \mapsto \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$. Let $p\colon R \to M_\infty(R)$ be a homomorphism mapping $1 \in R$ to a rank 1 projection. Then the homomorphism $id_A \otimes p$ induces weak equivalences of spectra $\mathbb{KK}(A \otimes_R M_\infty(R), B) \to \mathbb{KK}(A,B)$ and $\mathbb{KK}(B,A) \to \mathbb{KK}(B, A \otimes M_\infty(R))$.*

- *Let A be an R-algebra. Then we have a natural isomorphism $\pi_n \mathbb{KK}(R,A) \cong KH_n(A)$ for all n.*

$\square$

Because of the last part of the above result, we write $\mathbb{KH}(A) = \mathbb{KK}(R,A)$ for an R-algebra A. We also write $M_\infty(A) = A \otimes_R M_\infty(R)$.

We call an algebra homomorphism $\alpha\colon A \to B$ a *stable algebraic homotopy equivalence* if there is an algebra homomorphism $\beta\colon B \otimes_R M_\infty(R) \to A \otimes_R M_\infty(R)$ such that $\beta \circ (\alpha \otimes id_{M_\infty(R)}) \simeq id_{M_\infty(A)}$ and $(\alpha \otimes id_{M_\infty(R)}) \circ \beta \simeq id_{M_\infty(B)}$. By the above theorem, given an R-algebra C, a stable algebraic homotopy equivalence $\alpha\colon A \to B$ induces an equivalence of spectra $\alpha_*\colon \mathbb{KH}(A) \to \mathbb{KH}(B)$.

Now, we would like a version of homotopy algebraic K-theory associated to a coarse space and a fixed R-algebra, D, creating a coarsely excisive functor with 'coefficients' in the spectrum $\mathbb{KK}(D,R)$. In order to do this, we need to extend the above from algebras to algebroids.

To be more precise about what we need, given a ring R, recall (see [16]) that a category $\mathcal{A}$ is termed a *unital R-algebroid* if each morphism set $Hom(a,b)_{\mathcal{A}}$ is a left R-module, and composition of morphisms

$$Hom(b,c)_{\mathcal{A}} \times Hom(a,b)_{\mathcal{A}} \to Hom(a,c)_{\mathcal{A}}$$

is R-bilinear.

Non-unital R-algebroids are defined similarly, but we drop the requirement that identity morphisms $1 \in Hom(a,a)_{\mathcal{A}}$ have to exist.

A *unital algebroid homomorphism* between R-algebroids $\mathcal{A}$ and $\mathcal{B}$ is simply a functor $\phi\colon \mathcal{A} \to \mathcal{B}$ that is linear on each morphism set. Non-unital algebroid homomorphisms are defined similarly, but we drop the condition $\phi(1) = 1$ from the definition of a functor.

Given objects, a and b in an R-algebroid $\mathcal{A}$, an object $a \oplus b$ is called a *biproduct* of the objects a and b if it comes equipped with morphisms $i_a\colon a \to a \oplus b$, $i_b\colon b \to a \oplus b$, $p_a\colon a \oplus b \to a$, and $p_b\colon a \oplus b \to b$ satisfying the equations

$$p_a i_a = 1_a \qquad p_b i_b = 1_b \qquad i_a p_a + i_b p_b = 1_{a\oplus b}$$

An R-algebroid $\mathcal{A}$ is called *additive* if every pair of objects has a biproduct, and we have a *zero object* $0 \in Ob(\mathcal{A})$ such that $a \oplus 0$ is isomorphic to a for all $a \in Ob(\mathcal{A})$.

Now, in [11], Joachim defined the K-theory of C^*-categories, and in particular K-theory spectra, by use of a suitable functor from C^*-categories to C^*-algebras; C^*-algebra K-theory is of course well-known (see for instance [23, 25] for expositions). We adapt Joachim's approach here.

Firstly, given an algebroid $\mathcal{A}$, note that we can define an algebra $S(\mathcal{A})$ by writing

$$S(\mathcal{A}) = \oplus_{a,b\in Ob(\mathcal{A})} Hom(a,b)_{\mathcal{A}}.$$

Given $x \in Hom(a,b)_{\mathcal{A}}$ and $y \in Hom(c,d)_{\mathcal{A}}$, we define the product xy to be the composition $x \circ y \in Hom(c,b)_{\mathcal{A}}$ if $d = a$, and 0 otherwise.

In some sense, the algebra $S(\mathcal{A})$ obviously contains the information we want for K-theory purposes from the algebroid $\mathcal{A}$; this idea is made more concrete in [11] by looking at modules (and incidentally, Joachim's K-theory for C^*-categories is the same as that defined in [18]). However, it has the drawback of not being functorial; as in [11], the construction needs to be modified.

The algebra $S(\mathcal{A})$ is equipped with an obvious algebroid homomorphism $\eta\colon \mathcal{A} \to S(\mathcal{A})$ with the following universal property.

Lemma 3.2 *Let B be an algebra, and let $\phi\colon \mathcal{A} \to B$ be an algebroid homomorphism such that $\phi(x)\phi(y) = 0$ if x and y are non-composable morphisms in $\mathcal{A}$. Then there is a unique algebra homomorphim $\theta\colon S(\mathcal{A}) \to B$ such that $\phi = \theta \circ \eta$.*
□

Definition 3.3 Let $\mathcal{A}$ be an algebroid. We define the *Joachim algebra* $F(\mathcal{A})$ to be the free algebra generated by the morphisms of $\mathcal{A}$, with the relations

$$r(x) + s(y) = (rx + sy) \qquad r, s \in R, x, y \in Hom(a,b)_{\mathcal{A}},$$

and

$$(x)(y) = (xy) \qquad x \in Hom(b,c)_{\mathcal{A}},\ y \in Hom(a,b)_{\mathcal{A}}.$$

Here, we write (x) to denote the image in $F(\mathcal{A})$ of a morphism, x, in the algebroid $\mathcal{A}$.

Given an algebroid homomorphism $\phi\colon \mathcal{A} \to \mathcal{B}$, we have an induced algebra homomorphism $\phi_*\colon F(\mathcal{A}) \to F(\mathcal{B})$ defined by the obvious formula $\phi_*((x)) = (\phi(x))$ for each morphism x in $\mathcal{A}$. With such induced morphisms, F is a functor. Further, we have a natural algebroid homomorphism $\tau\colon \mathcal{A} \to F(\mathcal{A})$ defined by writing $\tau(x) = (x)$ for any morphism x, with the following universal property.

Lemma 3.4 *Let B be an algebra, and let $\phi\colon \mathcal{A} \to B$ be an algebroid homomorphism. Then there is a unique algebra homomorphim $\theta\colon S(\mathcal{A}) \to B$ such that $\phi = \theta \circ \tau$.* □

In particular, from the homomorphism $\eta\colon \mathcal{A} \to S(\mathcal{A})$, we obtain a unique algebra homomorphism $\sigma\colon F(\mathcal{A}) \to S(\mathcal{A})$ such that $\eta = \sigma \circ \tau$.

Proposition 3.5 *The map $\sigma\colon F(\mathcal{A}) \to S(\mathcal{A})$ is a stable algebraic homotopy equivalence.*

Proof: First of all, let $Ob(\mathcal{A})$ be countable, and let $Ob(\mathcal{A})_+ = Ob(\mathcal{A}) \cup \{e\}$ where e is just some element not in $Ob(\mathcal{A})$. Then the rows an columns of matrices in $M_\infty(R)$ can be considered to be indexed by the elements of $Ob(\mathcal{A})_+$, and for a pair $(a,b) \in Ob(\mathcal{A})_+ \times Ob(\mathcal{A})_+$, we have an element $\Theta_{a,b} \in M_\infty(R)$ with a 1 in column a, row b, and 0s everywhere else.

Defeine an algebroid homomorphism $\phi\colon \mathcal{A} \to F(\mathcal{A}) \otimes_R M_\infty(R)$ by the formula

$$\phi(x) = (x) \otimes \Theta_{a,b} \qquad x \in Hom(a,b)_{\mathcal{A}}.$$

Observe that $\phi(x)\phi(y) = 0$ if x and y are not composable. Hence, by the universal property of $S(\mathcal{A})$, we have a unique homomorphism $\theta\colon S(\mathcal{A}) \to F(\mathcal{A}) \otimes_R M_\infty(R)$ such that $\phi = \theta \circ \eta$.

Let $a \in Ob(\mathcal{A})$, and define

$$W_a(t) = \underset{a}{id} \otimes \begin{pmatrix} (1-t^2)\Theta_{e,e} & (t^3-2t)\Theta_{e,a} \\ -t\Theta_{a,e} & (1-t^2)\Theta_{a,a} \end{pmatrix}.$$

Then $W_a(t)$ is invertible, and we can define $H\colon \mathcal{A} \to (F(\mathcal{A}) \otimes_R M_\infty(R))[t]$ by the formula

$$H(x)(t) = W_b(t)(x \otimes \Theta_{a,b})W_a(t)^{-1} \qquad x \in Hom(a,b)_{\mathcal{A}}.$$

By the universal property of $F(\mathcal{A})$, we have $\tilde{H}\colon F(\mathcal{A}) \to (F(\mathcal{A}) \otimes_R M_\infty(R))[t]$ such that $H = \tilde{H} \circ \tau$. Observe that

$$\partial_0 \circ \tilde{H} = \theta(\sigma \otimes \underset{M_\infty(R)}{id}) \qquad \partial_1 \circ \tilde{H} = \underset{F(\mathcal{A})}{id} \otimes p$$

where $p\colon R \to M_\infty(R)$ is a homomorphism mapping $1 \in R$ to a rank 1 projection.

Thus $\theta \circ (\sigma \otimes id_{M_\infty(R)})$ and $id_{F(A)} \otimes p$ are algebraically homotopic. Similarly, $(\sigma \otimes id_{M_\infty(R)}) \circ \theta$ and $id_{S(\mathcal{A})} \otimes p$ are algebraically homotopic.

So the result holds if $Ob(\mathcal{A})$ is countable. A similar argument establishes the result when $Ob(\mathcal{A})$ is finite. So we need to worry about what happens when $Ob(\mathcal{A})$ is uncountable. In this case, we have

$$\mathcal{A} = \operatorname*{colim}_{\mathcal{A}' \subseteq \mathcal{A}} \mathcal{A}'$$

where $\mathcal{A}'$ is a full countable subcategory of $\mathcal{A}$.

Now, $F(\mathcal{A})$, although not functorial in general, is functorial with respect to inclusions of algebroids. Thus by the relevant universal properties

$$S(\mathcal{A}) = \operatorname*{colim}_{\mathcal{A}' \subseteq \mathcal{A}} S(\mathcal{A}')$$

and

$$F(\mathcal{A}) = \operatorname*{colim}_{\mathcal{A}' \subseteq \mathcal{A}} F(\mathcal{A}').$$

The result now follows. □

Definition 3.6 Let $\mathcal{A}$ and $\mathcal{B}$ be R-algebroids. Then we define the *bivariant algebraic K-theory spectrum*

$$\mathbb{KK}(\mathcal{A}, \mathcal{B}) = \mathbb{KH}(F(\mathcal{A}), F(\mathcal{B})).$$

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be R-algebroids with the same set of objects. We call a sequence of algebroid homomorphisms

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0$$

a *split short exact sequence* if i and j are the identity map on the set of objects, there is an algebroid homomorphism $k\colon \mathcal{C} \to \mathcal{B}$ such that $j \circ k = id_{\mathcal{C}}$, and for all objects a and b we have a split exact sequence of abelian groups

$$0 \to Hom(a,b)_{\mathcal{A}} \xrightarrow{i} Hom(a,b)_{\mathcal{B}} \xrightarrow{j} Hom(a,b)_{\mathcal{C}} \to 0$$

with splitting given by k.

It is easy to check that the functor F takes split short exact sequences of algebroids to split short exact sequences of algebras. We therefore have the following result.

Proposition 3.7 *Let*

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0$$

be a split exact short exact sequence of R-algebroids. Let D be a fixed R-algebra. Then we have induced homotopy fibrations of spectra

$$\mathbb{KK}(D, \mathcal{A}) \stackrel{i_*}{\to} \mathbb{KK}(D, \mathcal{B}) \stackrel{j_*}{\to} \mathbb{KK}(D, \mathcal{C})$$

and

$$\mathbb{KK}(\mathcal{C}, D) \stackrel{j^*}{\to} \mathbb{KK}(\mathcal{B}, D) \stackrel{i^*}{\to} \mathbb{KK}(\mathcal{A}, D).$$

□

We call an algebroid homomorphism $\phi\colon \mathcal{A} \to \mathcal{B}$ an *algebroid equivalence* if there is an algebroid homomorphism $\psi\colon \mathcal{B} \to \mathcal{A}$, along with natural isomorphisms $X : \psi \circ \phi \to id_{\mathcal{A}}$ and $Y : \phi \circ \psi \to id_{\mathcal{B}}$

Proposition 3.8 *Let $\phi\colon \mathcal{A} \to \mathcal{B}$ be an algebroid equivalence. Let D be an R-algebra. Then the induced map $\phi_*\colon \mathbb{KK}(D, \mathcal{A}) \to \mathbb{KK}(D, \mathcal{B})$ is a weak equivalence.*

Proof: It is clear that the algebras $S(\mathcal{A})$ and $S(\mathcal{B})$ are stably algebraically homotopy equivalent. It follows by proposition 3.5 that the map $\phi_* : F(\mathcal{A}) \to F(\mathcal{B})$ is a stable homotopy equivalence. The result now follows by stable homotopy invariance of bivariant algebraic K-theory. □

This innocuous-seeming result is vital to us when we look at assembly maps, and, incidentally, is the point where we need the matrix stability of bivariant algebraic K-theory.

4 Equivariant Coarse Assembly

Let G be a discrete group. A coarse space equipped with a right G-action is termed a *coarse G-space.*

We call a subset, A, of a coarse G-space X *cobounded* if there is a bounded subset $B \subseteq X$ such that $A \subseteq BG$. The *coarse G-category* is the category where the objects are coarse G-spaces, and the morphisms are controlled equivariant maps where the inverse image of a cobounded set is cobounded.

As in [20], the notions of coarse homotopy and flasqueness generalise in an obvious way to the equivariant notions of *coarse G-homotopy* and *G-flasqueness.*

Definition 4.1 We call a functor E_G from the coarse G-category to the category of spectra *coarsely G-excisive* if the following conditions hold.

- The spectrum $E_G(X)$ is weakly contractible whenever the coarse space X is G-flasque.
- The functor E_G takes coarse G-homotopy equivalences to weak homotopy equivalences of spectra.

- Given a coarsely excisive decomposition $X = A \cup B$, where A and B are coarse G-spaces, we have a homotopy push-out diagram

$$\begin{array}{ccc} E_G(A \cap B) & \to & E_G(A) \\ \downarrow & & \downarrow \\ E_G(B) & \to & E_G(X) \end{array} .$$

- Let X be a cobounded coarse G-space. Then the constant map $c\colon X \to +$ induces a stable equivalence $c_*\colon E_G(X) \to E_G(+)$.

As in the non-equivariant case, if E_G is coarsely G-excisive, we have a weak fibration

$$E_G(X) \stackrel{j}{\to} E_G(\mathcal{C}X) \stackrel{v_G}{\to} E_G(\mathcal{O}X)$$

and associated boundary map

$$\partial_G\colon \Omega E_G(\mathcal{O}X) \to E_G(X).$$

This map is termed the *equivariant assembly map* associated to the functor E_G.

Definition 4.2 Let E be a coarsely excisive functor. Then we say a coarsely G-excisive functor E_G has the *local property* relative to E if there is a natural map $i\colon E_G(X) \to E(X)$, such that if $X = \mathcal{O}Y$, where Y is a free coarse cocompact G-space, and $\pi\colon X \to X/G$ is the quotient map, then the composite

$$\pi_* \circ i = i \circ \pi_*\colon E_G(X) \to E(X/G)$$

is a stable equivalence.

In its most general terms, the main result of [20], which we apply to homotopy algebraic K-theory in the next section, is the following. We call it the *descent theorem.*

Theorem 4.3 *Let E_G be a coarsely G-excisive functor. Let E be a coarsely excisive functor with the local property relative to E_G. Let X be a free coarse G-space, that is, as a topological space, G-homotopy equivalent to a finite G-CW-complex.*

Suppose the coarse assembly map for the functor E and space X is a weak equivalence. Then the map $\partial_G\colon \Omega E_G(\mathcal{O}X) \to E_G(X)$ is injective at the level of stable homotopy groups. □

5 Homology

Our plan in this section is to define a coarsely excisive functor with 'coefficients' in algebraic KK-theory. As in [20], we adapt the approach taken for controlled algebraic K-theory in such articles as [1, 2, 4, 27].

Definition 5.1 Let X be a coarse space, and let $\mathcal{A}$ be an additive R-algebroid. Let $B(X)$ be the collection of bounded subsets of X, partially ordered by inclusion. Note that a partially ordered set can be regarded as a category; a *geometric $\mathcal{A}$-module* over X is a functor, M, from the collection of bounded subsets of X[3] to the category $\mathcal{A}$ such that for any bounded set B the natural map

$$\oplus_{x \in B} M(\{x\}) \to M(B)$$

induced by the various inclusions is an isomorphism, and the *support*

$$supp(M) = \{x \in X | \ M(\{x\}) \neq 0\}$$

has finite intersection with every bounded subset of X.

We call a subset $S \subseteq X$ *locally finite* if $S \cap B$ is finite whenever B is bounded. Thus the second of the above conditions says simply that $supp(M)$ is locally finite.

A *morphism* $\phi \colon M \to N$ between geometric $\mathcal{A}$-modules over X is a collection of morphisms $\phi_{x,y} \colon M_y \to N_x$ in the algebroid $\mathcal{A}$ such that for each fixed point $x \in X$, the morphism $\phi_{x,y}$ is non-zero for only finitely many points $y \in X$, and for each fixed point $y \in X$, the morphism $\phi_{x,y}$ is non-zero for only finitely many points $x \in X$.

Composition of morphisms $\phi \colon M \to N$ and $\psi \colon N \to P$ is defined by the formula

$$(\psi \circ \phi)_{x,y}(\eta) = \sum_{z \in X} \psi_{x,z} \circ \phi_{z,y}(\eta).$$

We define the *support* of a morphism ϕ

$$supp(\phi) = \{(x, y) \in X \times X \mid \phi_{x,y} \neq 0\}.$$

Definition 5.2 The category $\mathcal{A}[X]$ consists of all geometric $\mathcal{A}$-modules over X and morphisms such that the support is controlled with respect to the coarse structure of X.

Observe that $\mathcal{A}[X]$ is again an additive R-algebroid. Given a coarse map $f \colon X \to Y$, and a geometric $\mathcal{A}$-module, M, over X, we have a geometric $\mathcal{A}$-module $f_*[M]$ defined by writing $f_*[M](B) = M(f^{-1}[B])$ for each bounded set $B \subseteq Y$. Given a morphism $\phi \colon M \to N$ be a morphism in the category $\mathcal{A}[X]$, we have an induced morphism $f_*[\phi] \colon f_*[M] \to f_*[N]$ given by the formula

$$f_*[\phi]_{y_1,y_2} = \sum_{\substack{x_1 \in f^{-1}(y_1) \\ x_2 \in f^{-1}(y_2)}} \phi_{x_1,x_2}.$$

The above turns the assignment $X \mapsto \mathcal{A}[X]$ into a functor from the coarse category to the category of small R-algebroids and algebroid homomorphisms.

Now, the following is proved in exactly the same way as theorem 6.13 from [20]. Proposition 3.8 is needed to establish coarse homotopy invariance, and proposition 3.7 is needed for homotopy push-outs.

[3] Regarded as a category by looking at the usual partial ordering.

Theorem 5.3 *Let D be an R-algebra, and let $\mathcal{A}$ be an additive R-algebroid. Then the functor $X \mapsto \mathbb{KK}(D, \mathcal{A}[X])$ is coarsely excisive.* □

We also have an equivariant version of the construction. To be more precise, let X be a coarse G-space, let R be a ring, and let $\mathcal{A}$ be an additive R-algebroid. Then we call a geometric $\mathcal{A}$-module, M, over X *G-invariant* if $M_{xg} = M_x$ for all $x \in X$ and $g \in G$. A morphism $\phi \colon M \to N$ between such modules is termed *G-invariant* if $\phi_{xg,yg} = \phi_{x,y}$ for all $x, y \in X$.

Definition 5.4 We write $\mathcal{A}_G[X]$ to denote the category of G-invariant geometric $\mathcal{A}$-modules over X, and G-invariant morphisms.

The following result is similar to theorem 5.3.

Theorem 5.5 *The assignment $X \mapsto \mathbb{KK}(D, \mathcal{A}_G[X])$ is a coarsely G-excisive functor. Further, the functor $X \mapsto \mathbb{KK}(D, \mathcal{A}_G[X])$ has the local property relative to the functor $\mathbb{KK}(D, \mathcal{A}[X])$.* □

Now, we turn our attention to the associated isomorphism conjecture; the KH-isomorphism conjecture, as formulated and discussed in [3], is a special case.

Fix an R-algebra D, an R-algebroid $\mathcal{A}$, and consider the functor defined on cocompact topological spaces (with a coarse structure making any such space cobounded) by the formula

$$F_G(X) = \Omega \mathbb{KK}(D, \mathcal{A}_G[\mathcal{O}X]).$$

Then the *KK-assembly map* with coefficients in $\mathcal{A}$ is defined to be the map $c \colon F_G(X) \to F_G(+)$ induced by the constant map $X \to +$. Exactly as in theorem 8.17 of [20], this map amounts to the same thing (up to weak equivalence) as the equivariant assembly map

$$\partial_G \colon \Omega \mathbb{KK}(D, \mathcal{A}_G[\mathcal{O}X]) \to \mathbb{KK}(D, \mathcal{A}_G[X])$$

whenever G acts cocompactly on X.

Now, let $\underline{E}G$ be the *classifying space for proper actions* of G. As in [6], we describe $\underline{E}G$ as a unique (up to G-homotopy) G-CW-complex $E(G, \mathcal{F})$, with the property that, for as subgroup H, the fixed point set $\underline{E}G^H$ is empty if H is infinite, and G-contractible if H is finite.

Definition 5.6 The *KK-isomorphism conjecture* for the group G with coefficients in D and $\mathcal{A}$ asserts that the above KK-assembly map is a weak equivalence when $X = \underline{E}G$.

Now, the descent theorem can be applied to tell us things about injectivity of the KK-assembly map. To be precise, by theorem 5.7 and the above, we have the following result.

Theorem 5.7 *Suppose that the space $\underline{E}G$ is G-homotopy equivalent to a finite G-CW-complex. Suppose we have a coarse structure on $\underline{E}G$ where the G-action on $\underline{E}G$ is by coarse maps, and the coarse assembly map $\partial\colon \mathbb{KK}(D, \mathcal{A}[\mathcal{O}X]) \to \mathbb{KK}(D, \mathcal{A}[X]$ is a weak equivalence. Then the KK-assembly map is injective at the level of stable homotopy groups for the space $\underline{E}G$.* □

By theorem 2.17, we have the following corollary.

Corollary 5.8 *Suppose that the space $\underline{E}G$ is G-homotopy equivalent to a finite G-CW-complex. Suppose we have a coarse structure on $\underline{E}G$ where the G-action on $\underline{E}G$ is by coarse maps, and $\underline{E}G$ is coarsely homotopy equivalent to a finite coarse CW-complex. Then the KK-assembly map is injective at the level of stable homotopy groups for the space $\underline{E}G$.* □

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